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THE REPRESENTATION GROUPS AND PROJECTIVE REPRESENTATIONS OF THE POINT GROUPS AND THEIR APPLICATIONS

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CONTENTS

The different representation groups of the point groups are established and their character tables presented. These enable one to construct equivalent alternative sets of projective representations, as well as to provide an easy route to the determination of double group and space group representations. It is shown that these are uniquely determined, independent of the choice of representation group, but the availability of alternative representation groups allows greater scope for the processes of ascent and descent in symmetry, which are quite restricted in the context of projective representations.

1. Introduction

Recently (Döring 1956; Hurley 1966; Bradley & Cracknell 1972; Janssen 1973; Mozyrzymas 1975), interest has been shown in the projective representations of the point groups because of their value in facilitating the determination of the representations of the non-symmorphic space groups. We have found, however, that due to theoretical ambiguities in some of the methods used, none of the sets of character tables published so far is error-free. We have also found that the character systems of the projective representations are not always unique and have investigated

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the applicability of the different possibilities. We shall follow a logical approach based on Schur's original prescription which fully investigates the representation groups rather than choosing suitable factor systems.

2. Representation groups and multiplicators

Schur (1904) defined a representation group, $\mathscr{R}$, of a group G, as an abstract group possessing an invariant subgroup, called the multiplicator M, which is contained in both the centre, Z, and the commutator subgroup, $\mathscr{K}$, of $\mathscr{R}$ such that the factor group $\mathscr{R}/M$ is isomorphic to G and the order of M is as small as possible without being trivial, unless no non-trivial possibilities exist. The order of $\mathscr{R}$ is therefore the product of the orders of G and M. The mapping of $\mathscr{R}$ onto $\mathscr{R}/M$ is a canonical epimorphism with kernel M and image $\mathscr{R}/M$, since it maps the elements of $\mathscr{R}$ onto the elements of a group whose elements are cosets. Since $\mathscr{R}/M$ is isomorphic to G, there is an epimorphism, π, from $\mathscr{R}$ onto G.

A representation group is therefore a central extension of M by G. It is not necessarily unique although M is unique for a given group G. It cannot be a supergroup of G and hence cannot be written as a direct or semi-direct product structure involving G and M.

If one extends the concept of a representation of a group G of elements $\{g_i\}$ to allow a multiplication law for the representative matrices, δ, of the form

$$\delta(g_i)\,\delta(g_j) = \omega(g_i, g_j)\,\delta(g_i g_j)$$

where the factor systems $\omega(g_i, g_j)$ are complex numbers of unit modulus, then it can be shown by the following argument that the true (or vector) representations of $\mathscr{R}$ correspond to either vector or generalized (or projective, or ray) representations of G. A representative matrix $\Delta(r_i)$ of $\mathscr{R}$ is also a representative matrix $\delta(\pi r_i)$ of G since the epimorphism π maps the element r_i of $\mathscr{R}$ onto the element πr_i of G. Since Δ is a true representation of $\mathscr{R}$, the product of the representative matrices of two elements,

$$\Delta(r_i)\,\Delta(r_j) = \Delta(r_i r_j),$$

the representative matrix of the product of the elements. But we also have

$$\begin{aligned}\Delta(r_i)\,\Delta(r_j) &= \delta(\pi r_i)\,\delta(\pi r_j)\\ &= \delta(\pi r_i r_j),\end{aligned}$$

the representative matrix of an element of G. Hence

$$\Delta(r_i r_j) = \delta(\pi r_i r_j)$$

and therefore Δ is also a representation of G.

Now let r_k be that element of $\mathscr{R}$ such that $\pi r_k = g_k$. Because the mapping of the product of two elements,

$$\pi(r_k r_l) = (\pi r_k)\,(\pi r_l) = g_k g_l = g_{kl} = \pi r_{kl},$$

the mapping of another element, it follows that $\pi(r_{kl} r_k^{-1} r_l^{-1}) = e$, the identity of G. This is satisfied if

$$r_k r_l = m_{kl}\, r_{kl},$$

where m_{kl} is an element of M which lies in the commutator subgroup K of G and which commutes with all elements of $\mathscr{R}$. Hence the representative matrices of M must commute with all representative matrices of $\mathscr{R}$ in a given irreducible representation, Δ, and hence by Schur's lemma must be multiples, ω, of the unit matrix. Hence the product of two representative matrices,

$$\Delta(r_i)\,\Delta(r_j) = \Delta(r_i r_j) = \Delta(m_{ij} r_{ij}) = \omega(r_i r_j)\,\Delta(r_{ij}),$$

a unit multiple of another representative matrix, and since

$$\Delta(r_i) = \delta(g_i),$$

we have

$$\delta(g_i)\,\delta(g_j) = \omega(g_i, g_j)\,\delta(g_i g_j)$$

and therefore δ is a projective representation of G.

Two projective representations, δ and δ', are said to be associated if $\delta(g_i) = u(g_i)\,\delta'(g_i)$, where $u(g_i) \neq 0$ is a complex number of unit modulus. To these correspond associated factor systems ω and ω' which together with all other factor systems associated to them form a multiplicative Abelian group, $B^2(G)$ of associated factor systems. This is an invariant subgroup of the group of all factor systems $Z^2(G)$. The factor group $Z^2(G)/B^2(G)$ is isomorphic to $H^2(G)$, the group of all classes of associated factor systems which is in this context precisely the multiplicator M. In cohomology theory, the factor systems are those two-dimensional co-chains which are two-dimensional co-cycles, the sets of associated factor systems are those two-dimensional co-chains which are also co-boundaries of some one-dimensional co-chains and $H^2(G)$ is the second cohomology group of extensions of G by M.

3. Determination of the multiplicators

The multiplicators of the point groups are most efficiently determined by an *aufbau* process starting with the cyclic groups, namely C_n, S_{2n} and $C_{(2n-1)h}$. These are single generator groups and are hence Abelian. Their representation groups are hence also single generator groups, also Abelian and therefore have commutator subgroups, C_1. Since the multiplicator must be contained in the commutator subgroups of the representation groups, the multiplicators of the cyclic groups must all be C_1 and therefore the representation group coincides with the original group and there are no projective representations.

The multiplicators of the dihedral groups D_{2n+1} of order $4n+2$ and hence also $C_{(2n+1)v}$ ($\cong D_{2n+1}$) may be determined by theorem v of Schur (1907). This is because all of their Sylow subgroups are cyclic and hence the order of their multiplicator is divisible by no prime number greater than 1. Their multiplicator is hence C_1.

For groups of the family D_{4n}($\cong C_{4nv} \cong D_{2nd}$), non-trivial multiplicators can be found and it will be sufficient to show that one representation group of twice their order exists to prove that the multiplicators are all C_2. The double groups D'_{4n} are known to have the property $D'_{4n}/C'_1 \cong D_{4n}$ since they are central extensions of C'_1 by D_{4n} and since their commutator is C'_n, C'_1($\cong C_2$) is a possible multiplicator. Since this group is of the minimal non-trivial order, the multiplicator must be isomorphic to the abstract group C_2 for *all* possible representation groups.

The *Vierergruppe*, D_2($\cong C_{2v} \cong C_{2h}$) will be the first example of a direct product group. To apply theorem vi of Schur (1907), D_2 is factorized as $C_2 \times C_2$ and the quotient group is formed of each factor with its own commutator subgroup, namely $C_2/C_1 \cong C_2$ for each factor. The orders of these quotient groups are then factorized into primes and the highest common factors (hcf) of all possible pairs of prime factors corresponding to different quotient groups are multiplied together.

The multiplicator of D_2 is then given as

$$\begin{aligned} M(D_2) &\cong M(C_2)\times M(C_2)\times C_{\mathrm{hcf}(2,2)} \\ &\cong C_1\times C_1\times C_2 \\ &\cong C_2. \end{aligned}$$

The multiplicators of the tetrahedral, octahedral and icosahedral rotational groups, T, O and I respectively, may now be determined by theorem v of Schur (1907) since, apart from cyclic groups of odd order, their Sylow subgroups are respectively D_2, D_4 and D_2. These all have multiplicators isomorphic to C_2 and hence if T, O and I are to have non-trivial multiplicators, these must all be isomorphic to C_2. The multiplicator of the regular tetrahedral group, T_d, must also be isomorphic to C_2 since T_d is isomorphic to O.

All remaining point groups can be regarded as direct product groups:

$$\begin{aligned} C_{2nh} &\cong C_{2n}\times C_2, \\ D_{4n+2} &\cong C_{(4n+2)v} \cong D_{(2n+1)d} \cong D_{(2n+1)h} \cong D_{2n+1}\times C_2, \\ D_{2nh} &\cong D_{2n}\times C_2, \\ T_h &\cong T\times C_2, \\ O_h &\cong O\times C_2, \\ I_h &\cong I\times C_2, \end{aligned}$$

and hence their multiplicators can be determined using theorem vi of Schur (1907).

Finally the spherical rotation group, K, is known to have a double group, K' such that

$$K'/C_1' \cong K.$$

This obeys the requirements for a representation group and hence the multiplicator is determined to be isomorphic to C_2. Further, since the commutator of K is K, the double group is the only representation group of K, in accordance with theorem ii of Schur (1907). The spherical group relevant to atoms is $K_h = K\times S_2$ and contains reflexion planes and the inversion. This is a direct product group and hence by theorem vi of Schur (1907) its multiplicator is also isomorphic to C_2.

The use of the above determination of the multiplicator as a means of finding the second cohomology group is a labour-saving method for those problems involving the extension of a group by its multiplicator and is far simpler than direct application of cohomology theory.

The results may be summarized as follows.

multiplicator isomorphic to	point groups
C_1	C_n, S_{2n}, $C_{(2n-1)h}$, D_{2n+1}, $C_{(2n+1)v}$
C_2	C_{2nh}, C_{2nv}, D_{2n}, D_{nd}, $D_{(2n+1)h}$, T, T_d, T_h, O, I, I_h, K, K_h
$C_2\times C_2$	O_h
$C_2\times C_2\times C_2$	D_{2nh}

It might be mentioned that although it is the case for the point groups that the multiplicators are isomorphic to C_1 or products of C_2, multiplicators of other types can appear, e.g. if p is a prime number, the multiplicator of the direct product group $C_p\times C_p$ (used in describing molecules exhibiting internal rotation) is isomorphic to C_p.

4. Determination of the representation groups

The determination of the representation groups is usefully preceded by the determination of the maximum possible number of such groups using theorem I and II of Schur (1907). To apply these theorems we need to know the multiplicators, M, determined in the preceding section and the commutator subgroups, K, of the point groups, G, themselves. The quotient groups G/K, which are necessarily Abelian, are then factorized in terms of cyclic groups $C_{\epsilon_1} \times C_{\epsilon_2} \times C_{\epsilon_3} \times \ldots$, where the orders ϵ_1, ϵ_2, … are the integers referred to by Schur as the invariants of the quotient group. The multiplicator is likewise factorized and its invariants may be denoted as $e_1, e_2, \ldots$. Schur (theorem I) then proved that an upper bound to the number of representation groups, $n_{\max}$, was given by the product of all possible highest common factors of the type hcf (ϵ_i, e_j). When G is a complete group, for example the groups T_d and O, this upper bound is the actual number, n. When $K = G$, as is the case for the point groups I and K, there can only be one representation group (theorem II of (1907) and theorem IV of (1904)) independent of the multiplicator. The results may be summarized in the following table:

Table 1. The commutator subgroups, multiplicators and numbers of representation groups of the point groups

G	K	G/K	M	$n_{\max}$	n
C_{2n-1}	C_1	C_{2n-1}	C_1	1	1
C_{2n}, S_{2n}, C_{nh} (n odd)	C_1	C_{2n}	C_1	1	1
D_{2n-1}, $C_{(2n-1)v}$	C_{2n-1}	C_2	C_1	1	1
C_{2nh}	C_1	$C_{2n} \times C_2$	C_2	4	2
D_{2n}, C_{2nv}, D_{nd}, D_{nh} (n odd)	C_n	$C_2 \times C_2$	C_2	4	$2 (n = 1)$; $3 (n \neq 1)$
D_{2nh}	C_n	$C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$	512	$1 (n = 1)$; $2 (n \neq 1)$
T	D_2	C_3	C_2	1	1
T_d, O	T	C_2	C_2	2	2
T_h	D_2	$C_2 \times C_3$	C_2	2	2
O_h	T	$C_2 \times C_2$	$C_2 \times C_2$	16	4
I	I	C_1	C_2	1	1
I_h	I	C_2	C_2	2	2
K	K	C_1	C_2	1	1
K_h	K	C_2	C_2	2	2

The determination of the actual number, n, of non-isomorphic representation groups of a given group, G, requires an examination of the $n_{\max}$ possibilities to see if they lead to groups and then what isomorphisms exist between them. This process can be facilitated by considering first the representation groups of groups which can be specified by two generators and then using these as a basis in a composition series for considering those groups which must be specified by three or four generators and then stepwise to those groups which are conveniently specified by four or five generators.

Let us consider a group G specified by two generators A and B such that $A^l = B^m = E$ and $BA = A^x B^y$. A representation group $\mathscr{R}$ for G must be specifiable in terms of two generators, P and Q such that $P^\lambda = Q^\mu = E$ and $QP = P^\xi Q^\eta$. The order of G is lm since for all point groups in question $A^{\frac{1}{2}l} = B^{\frac{1}{2}m}$ and hence the order required for $\mathscr{R}$ is $2lm$ since the multiplicator for all two-generator point groups is of order 2. Hence, if $P^{\frac{1}{2}\lambda} \neq Q^{\frac{1}{2}\mu}$, $2lm = \lambda\mu$ i.e. $\lambda = 2l$, $\mu = m$ or $\lambda = l$, $\mu = 2m$. If, however, $P^{\frac{1}{2}\lambda} = Q^{\frac{1}{2}\mu}$, then $2lm = \frac{1}{2}\lambda\mu$ i.e. $\lambda = 2l$, $\mu = 2m$. (Cases such as $\lambda = 4l$, $\mu = m$ are excluded since these would not correspond to a multiplicator of order 2.) Considering now the relation $BA = A^xB^y$, the corresponding relation $QP = P^\xi Q^\eta$ in the representation group can permit different combinations of values of ξ and η according to the values of λ and μ. The results can be summarized as follows:

label	generating relations in $\mathscr{R}$		multiplicator	commutators of $\mathscr{R}$: $\mathscr{K}\begin{pmatrix}x=1\\y=1\end{pmatrix}$	commutators of $\mathscr{R}$: $\mathscr{K}\begin{pmatrix}x=2n-1\\y=1\end{pmatrix}$
ρ_1	$P^{2l} = Q^m = E$	$QP = P^xQ^y$	$\{E, P^l\}$	E	n even: $P^{2l} = E$ n odd: $(P^4)^{\frac{1}{2}l} = E$
ρ_2	$P^{2l} = Q^m = E$	$QP = P^{x+l}Q^y$	$\{E, P^l\}$	$(P^2)^l = E$	$P^{2l} = E$
ρ_3	$P^l = Q^{2m} = E$	$QP = P^xQ^y$	$\{E, Q^m\}$	E	$P^l = E$
ρ_4	$P^l = Q^{2m} = E$	$QP = P^xQ^{y+m}$	$\{E, Q^m\}$	$(Q^2)^m = E$	n even: $(P^2Q^2)^l = E$ n odd: $P^l = (Q^2)^2 = E$; $Q^2P = P^{l-1}Q^2$
ρ_5	$P^{2l} = Q^{2m} = E$	$QP = P^xQ^y$	$\{E, P^l = Q^m\}$	E	n even: $P^{2l} = E$ n odd: $(P^4)^{\frac{1}{2}l} = E$
ρ_6	$P^{2l} = Q^{2m} = E$	$QP = P^{x+l}Q^y$	$\{E, P^l = Q^m\}$	$(P^2)^l = E$	$P^{2l} = E$
ρ_7	$P^{2l} = Q^{2m} = E$	$QP = P^xQ^{y+m}$	$\{E, P^l = Q^m\}$	$(P^2)^l = E$	$P^{2l} = E$
ρ_8	$P^{2l} = Q^{2m} = E$	$QP = P^{x+l}Q^{y+m}$	$\{E, P^l = Q^m\}$	E	n even: $P^{2l} = E$ n odd: $P^{2l-4} = E$

Of the eight possibilities it may be noted that $\rho_5 = \rho_8$ and $\rho_6 = \rho_7$ since for these groups the invariant element $P^l = Q^m$. Among the relevant point groups, we always have $y = 1$ and either $x = 1$ (for the C_{2nh} family) or $x = 2n-1$ (for the D_{2n} family). For these two cases, the generating relations of the commutator subgroups of the representation groups are listed. Comparison with the elements of the multiplicator shows that for the C_{2nh} groups, ρ_2 and ρ_6 are possible representation groups when l is even which is the case since $l = 2n$, and ρ_4 is a representation group when m is even, which is satisfied since $m = 2$ for the C_{2nh} point groups. In fact ρ_2 and ρ_6 are isomorphic since different choices of generators will lead to the two different formulations of the group. There are thus only two different representation groups for each group of the C_{2nh} family.

In the case of the D_{2n} groups, comparison of commutator subgroups and multiplicators shows that when n is even, ρ_1, ρ_2, ρ_5 and ρ_6 are possible representation groups while when n is odd, ρ_2, ρ_4 and ρ_6 are the possible representation groups. Detailed examination of the structure of these groups shows that when n is even, ρ_5 is isomorphic to ρ_1 and hence there will be three representation groups, albeit of different types, for each value of $n \neq 1$. When $n = 1$, $\rho_4 = \rho_2$ and so there are then only two non-isomorphic representation groups, namely ρ_2 and ρ_6.

This approach may be extended to the remaining point groups by considering the following composition series in which each group is a normal subgroup of the following group so that by addition of one generator and a specification of its multiplicative properties with the other generators, one can arrive at the next group in the series:

$$D_2 \to T \to T_h$$

$$D_2 \to T \to O(\cong T_d) \to O_h$$

$$D_2 \to T \to I \to I_h$$

The derivation of the representation groups for the cubic and icosahedral point groups was straightforward, even for the case of O_h where the multiplicator was of increased order. The groups of the family D_{2nh}, however, where the multiplicator is of order 8 required an approach similar to that for two generators. It was found that of Schur's 512 possibilities only 64 need be considered *a priori*, of which only 14 satisfied the conditions relating the commutators and multiplicators. Of these 14 possibilities it turned out that for any given group of the D_{2nh} $(n > 1)$ family, only two non-isomorphic representation groups could be found while for D_{2h} itself there was only one possible representation group.

It has already been mentioned in the determination of the multiplicators of the spherical groups that the double group K' is a representation group for the rotation group K and this must be the only such group. The double group K'_h is one representation group of K_h, the second being one in which a non-invariant four-fold element and its inverse map onto the inversion.

The actual numbers, n, of representation groups may be found collected in table 1.

5. Character tables of the representation groups

The following character tables of the representation groups are listed here for the first time. These supersede all previous compilations of projective representations, either because earlier tables do not list more than one possible set of projective representations (Döring 1956; Hurley 1966) or, additionally, they contain demonstrable errors (e.g. the D_{2h} tables of Janssen (1973) and Mozyrzymas (1975)), usually in an incorrect specification of the sign of some characters. The advantage of using the full representation group rather than a set of characters of the projective representations of the point group is that $\mathscr{R}$ is a genuine group and hence operations involving the projective representations, such as symmetrization of powers, can be performed without need for any additional algebraic formulations. The tables are also useful as they contain all central extensions of G by M and hence may assist in physical problems where group extensions are needed as well as enlarging the categories of abstract groups for which character tables are available.

The tables are presented in the format customary in molecular physics in which {1, 2, 3, 4, 5, 6, 8, 12}-dimensional representations are denoted by the letters {A, E, T, G, H, I, K, O} of the Mulliken-Placzek system irrespective of whether the degeneracy is separable (Frobenius & Schur 1906) or not. The complex conjugate components of separably degenerate representations have been denoted by the superscripts + and −. The elements of the multiplicator, M, have been placed at the beginning and, since they coincide with the centre of the representation group, their characters are ± those for the identity element. The vector representations have positive characters for all elements of the multiplicator, while the projective representations have half of these characters positive and half negative. The different classes of representations have been called ω-representations by Bradley & Backhouse (1970) and are denoted by subscripts α, β, $\alpha\beta$, etc. (except for those groups with multiplicator C_2 where the well-known double group is a representation group: in such cases the double-valued representations denoted by half-integral subscripts are the projective or α-representations). The elements of the representation group have been described in terms of generators P, Q, R, ... and the elements of the point group (described in terms of generators A, B, C, ...) to which these correspond are indicated in the relevant columns below the characters. The composition of a class has been denoted by a symbol of type $X\epsilon_x$ which means that it contains X elements of order x. The relations between the generators for both $\mathscr{R}$ and G have been collected on the right-hand side. Where feasible, inverse pairs of elements have been collected on the same horizontal line and, when in the same class, are separated by commas. Elements separated by semi-colons are not inverses.

The abstract generators of the point groups may be identified by means of table 2.

It may be mentioned that not only do these tables contain the first correct characters for the projective representations of D_{2h} but also they consider the icosahedral groups for the first time.

Table 2. Identification of the generating elements of the point groups with the abstract generators derived by mapping from the representation groups

G	A	B	C	D	F	I
C_{2nh}	C_{2n}	σ_h				
C_{2nv}	C_2	σ_v				
D_{2n}	C_{2n}	C_2'				
D_{nd}	S_{2n}	σ_d				
$D_{(2n+1)h}$	S_{2n+1}	σ_v				
D_{2nh}	C_{2n}	σ_v	σ_h			
T	C_2^z	C_2^x	C_3^{xyz}			
T_h	C_2^z	C_2^x	C_3^{xyz}			S_2
T_d	C_2^z	C_2^x	C_3^{xyz}	σ_d		
O	C_2^z	C_2^x	C_3^{xyz}	$C_2'^{zx}$		
O_h	C_2^z	C_2^x	C_3^{xyz}	$C_2'^{zx}$		S_2
I	C_2^z	C_2^x	C_3^{xyz}		$C_5^{(\Phi 0 \Phi^{-1})}$	
I_h	C_2^z	C_2^x	C_3^{xyz}		$C_5^{(\Phi 0 \Phi^{-1})}$	S_2

Table 3. The character tables of the representation groups of the point groups

			$1 \leqslant p \leqslant n$	$1 \leqslant p \leqslant n-1$	$1 \leqslant p \leqslant n-1$		$1 \leqslant p \leqslant 2n-1$	
	$1\epsilon_1$	$1\epsilon_2$	$2\epsilon_{2n/\mathrm{hcf}(n,2p-1)}$	$1\epsilon_{n/\mathrm{hcf}(n,p)}$	$1\epsilon_{\mathrm{lcm}(n,2)}$	$2\epsilon_4$	$2\epsilon_{\mathrm{lcm}(2n/[\mathrm{hcf}(2n,p)],4)}$	$8n$ elements
			$P^{2p-1}Q^2$			Q^3	P^pQ^3	$P^{2n}=Q^4=E$
$\mathscr{R}_1(C_{2nh})$	E	Q^2	P^{2p-1}	P^{2p}	$P^{2p}Q^2$	Q	P^pQ	$QP=PQ^3$
A_g	1	1	1	1	1	1	1	$\alpha=+1$
B_g	1	1	-1	1	1	$(-1)^n$	$(-1)^{n+p}$	
A_u	1	1	1	1	1	-1	-1	
B_u	1	1	-1	1	1	$(-1)^{n+1}$	$(-1)^{n+p+1}$	
$\leqslant l \leqslant n-1; E_{lg}$ E_{lg}^+	1	1	$e^{il(2p-1)\pi/n}$	$e^{2il p\pi/n}$	$e^{2il p\pi/n}$	$(-1)^l$	$-e^{il p\pi/n}$	
E_{lg}^-	1	1	$e^{-il(2p-1)\pi/n}$	$e^{-2il p\pi/n}$	$e^{-2il p\pi/n}$	$(-1)^l$	$-e^{-il p\pi/n}$	
$\leqslant l \leqslant n-1; E_{lu}$ E_{lu}^+	1	1	$e^{il(2p-1)\pi/n}$	$e^{2il p\pi/n}$	$e^{2il p\pi/n}$	$(-1)^{l+1}$	$e^{il p\pi/n}$	
E_{lu}^-	1	1	$e^{-il(2p-1)\pi/n}$	$e^{-2il p\pi/n}$	$e^{-2il p\pi/n}$	$(-1)^{l+1}$	$e^{-il p\pi/n}$	
ven; $E_{\frac{1}{2}n}$	2	-2	0	-2	2	0	0	$\alpha=-1$
E_n	2	-2	0	2	-2	0	0	
$\leqslant l \leqslant \frac{1}{2}(n-1); G_l$ G_l^+	2	-2	0	$2e^{2\pi li/n}$	$-2e^{2\pi li/n}$	0	0	
G_l^-	2	-2	0	$2e^{-2\pi li/n}$	$-2e^{-2\pi li/n}$	0	0	
C_{2nh}	E		A^{2p-1}	A^{2p}		B	A^pB	$A^{2n}=B^2=E$; $BA=AB$

			$1 \leqslant p \leqslant n$	$\begin{Bmatrix} 1 \leqslant p \leqslant n-1 \\ n+1 \leqslant p \leqslant 2n-1 \end{Bmatrix}$		$1 \leqslant p \leqslant 2n-1$	
	$1\epsilon_1$	$1\epsilon_2$	$2\epsilon_{4n/\mathrm{hcf}(n,2p-1)}$	$\epsilon_{2n/\mathrm{hcf}(2n,p)}$	$2\epsilon_4$	$2\epsilon_{\mathrm{lcm}(4n/[\mathrm{hcf}(4n,p)],4)}$	$8n$ elements
			$P^{2n+2p-1}$		$P^{2n}Q$	$P^{2n+p}Q$	$P^{4n}=Q^4=E$; $P^{2n}=Q^2$
$\mathscr{R}_2(C_{2nh})$	E	P^{2n}	P^{2p-1}	P^{2p}	Q	P^pQ	$QP=P^{2n+1}Q$
A_g	1	1	1	1	1	1	$\alpha=+1$
B_g	1	1	-1	1	$(-1)^n$	$(-1)^{n+p}$	
A_u	1	1	1	1	-1	-1	
B_u	1	1	-1	1	$(-1)^{n+1}$	$(-1)^{n+p+1}$	
$1 \leqslant l \leqslant n-1; E_{lg}$ E_{lg}^+	1	1	$e^{il(2p-1)\pi/n}$	$e^{2il p\pi/n}$	$(-1)^l$	$-e^{il p\pi/n}$	
E_{lg}^-	1	1	$e^{-il(2p-1)\pi/n}$	$e^{-2il p\pi/n}$	$(-1)^l$	$-e^{-il p\pi/n}$	
$1 \leqslant l \leqslant n-1; E_{lu}$ E_{lu}^+	1	1	$e^{il(2p-1)\pi/n}$	$e^{2il p\pi/n}$	$(-1)^{l+1}$	$e^{il p\pi/n}$	
E_{lu}^-	1	1	$e^{-il(2p-1)\pi/n}$	$e^{-2il p\pi/n}$	$(-1)^{l+1}$	$e^{-il p\pi/n}$	
n odd; E_α	2	-2	0	$2(-1)^p$	0	0	$\alpha=-1$
$1 \leqslant l \leqslant \frac{1}{2}n; G_{l\alpha}$ $G_{l\alpha}^+$	2	-2	0	$2e^{ip(2l-1)\pi/n}$	0	0	
$G_{l\alpha}^-$	2	-2	0	$2e^{-ip(2l-1)\pi/n}$	0	0	
C_{2nh}	E		A^{2p-1}	A^{2p}	B	A^pB	$A^{2n}=B^2=E$

TABLE 3 (cont.)

$\mathscr{R}_1(D_{2n})$	$1\epsilon_1$ E	$1\epsilon_2$ P^{2n}	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,\,p)}$ P^{4n-p} P^p	$2n\epsilon_4$ $0 \leqslant q \leqslant 2n-1$ $P^{2q}Q$	$2n\epsilon_4$ $0 \leqslant q \leqslant 2n-1$ $P^{2q+1}Q$	$8n$ elements $P^{4n} = Q^4 = E; P^{2n} = Q^2$ $QP = P^{4n-1}Q$
A_1	1	1	1	1	1	$\alpha = +1$
A_2	1	1	1	-1	-1	
B_1	1	1	$(-1)^p$	1	-1	
B_2	1	1	$(-1)^p$	-1	1	
$1 \leqslant l \leqslant n-1;\ E_l$	2	2	$2\cos(lp\pi/n)$	0	0	
$1 \leqslant l \leqslant n;\ E_{l\alpha}$	2	-2	$2\cos\{(2l-1)p\pi/2n\}$	0	0	$\alpha = -1$
D_{2n}	E		A^p	$A^{2q}B$ $0 \leqslant q \leqslant n-1$	$A^{2q+1}B$ $0 \leqslant q \leqslant n-1$	$A^{2n} = B^2 = E$ $BA = A^{2n-1}B$

$\mathscr{R}_2(D_{2n})$	$1\epsilon_1$ E	$1\epsilon_2$ P^{2n}	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,\,p)}$ P^{4n-p} P^p	$2n\epsilon_2$ $0 \leqslant q \leqslant 2n-1$ $P^{2q}Q$	$2n\epsilon_2$ $0 \leqslant q \leqslant 2n-1$ $P^{2q+1}Q$	$8n$ elements $P^{4n} = Q^2 = E$ $QP = P^{4n-1}Q$
A_1	1	1	1	1	1	$\alpha = +1$
A_2	1	1	1	-1	-1	
B_1	1	1	$(-1)^p$	1	-1	
B_2	1	1	$(-1)^p$	-1	1	
$1 \leqslant l \leqslant n-1;\ E_l$	2	2	$2\cos\{lp\pi/n\}$	0	0	
$1 \leqslant l \leqslant n;\ E_{l\alpha}$	2	-2	$2\cos\{(2l-1)p\pi/2n\}$	0	0	$\alpha = -1$
D_{2n}	E		A^p	$nA^{2q}B$ $0 \leqslant q \leqslant n-1$	$nA^{2q+1}B$ $0 \leqslant q \leqslant n-1$	$A^{2n} = B^2 = E$ $BA = A^{2n-1}B$

$\mathscr{R}_3(D_{4n})$	$1\epsilon_1$ E	$1\epsilon_2$ P^{4n}	$1 \leqslant p \leqslant 2n$ $2\epsilon_{8n/\mathrm{hcf}(4n,\,2p-1)}$ $P^{4n-2p+1}$ P^{2p-1}	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,\,p)}$ P^{8n-2p} P^{2p}	$4n\epsilon_2$ $0 \leqslant q \leqslant 4n$ $P^{2q}Q$	$4n\epsilon_4$ $0 \leqslant q \leqslant 4n$ $P^{2q+1}Q$	$16n$ elements $P^{8n} = Q^2 = E$ $QP = P^{4n-1}Q$
A_1	1	1	1	1	1	1	$\alpha = +1$
A_2	1	1	1	1	-1	-1	
B_1	1	1	-1	1	1	-1	
B_2	1	1	-1	1	-1	1	
$l \leqslant 2n-1;\ E_l$	2	2	$2\cos\{(2p-1)l\pi/2n\}$	$2\cos\{lp\pi/n\}$	0	0	
$\leqslant l \leqslant n;\ G_{l\alpha}$ $G^+_{l\alpha}$	2	-2	$2\mathrm{i}\sin\{(2p-1)(2l-1)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	0	0	$\alpha = -1$
$G^-_{l\alpha}$	2	-2	$-2\mathrm{i}\sin\{(2p-1)(2l-1)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	0	0	
D_{4n}	E		A^{2p-1} $A^{4n-2p+1}$	A^{2p}	$A^{2q}B$ $0 \leqslant q \leqslant 2n$	$A^{2q+1}B$ $0 \leqslant q \leqslant 2n$	$A^{4n} = B^2 = E$ $BA = A^{4n-1}B$

TABLE 3 (cont.)

		$1\epsilon_1$	$1\epsilon_2$	$1 \leqslant p \leqslant n-1$ $2\epsilon_{4n-2}$	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{(4n-2)/\text{hcf}(4n-2,\,p)}$	$1 \leqslant p \leqslant n-1$ $2\epsilon_{(2n-1)/\text{hcf}(2n-1,\,p}$	$(4n-2)\,\epsilon_4$ $0 \leqslant q \leqslant 2n-2$	$(4n-2)\,\epsilon_2$ $0 \leqslant q \leqslant 2n-2$	$(16n-8)$ elements
	$\mathscr{R}_3(D_{4n-2})$	E	Q^2	$P^{4n-2-2p}Q^2$ $P^{2p}Q^2$	$P^{4n-1-2p}Q^2$ P^{2p-1}	$P^{4n-2-2p}$ P^{2p}	$P^{2q}Q^3$ $P^{2q}Q$	$P^{2q+1}Q^3$ $P^{2q+1}Q$	$P^{4n-2} = Q^4 = E$ $QP = P^{4n-3}Q^3$
	A_1	1	1	1	1	1	1	1	$\alpha = +1$
	A_2	1	1	1	1	1	-1	-1	
	B_1	1	1	1	-1	1	1	-1	
	B_2	1	1	1	-1	1	-1	1	
$1 \leqslant l \leqslant 2n-2;$	E_l	2	2	$2\cos\dfrac{2lp\pi}{2n-1}$	$2\cos\dfrac{l(2p-1)\,\pi}{2n-1}$	$2\cos\dfrac{2lp\pi}{2n-1}$	0	0	
	E_α	2	-2	-2	0	2	0	0	$\alpha = -1$
$1 \leqslant l \leqslant n-1;\ G_{l\alpha}$	$G^+_{l\alpha}$	2	-2	$-2\cos\dfrac{4lp\pi}{2n-1}$	$2\mathrm{i}\sin\dfrac{2l(2p-1)\,\pi}{2n-1}$	$2\cos\dfrac{4lp\pi}{2n-1}$	0	0	
	$G^-_{l\alpha}$	2	-2	$-2\cos\dfrac{4lp\pi}{2n-1}$	$-2\mathrm{i}\sin\dfrac{2l(2p-1)\,\pi}{2n-1}$	$2\cos\dfrac{4lp\pi}{2n-1}$	0	0	
	D_{4n-2}	E			A^{2p-1}	A^{2p} $A^{4n-2-2p}$	$(2n-1)\,A^{2q}B$	$(2n-1)\,A^{2q+1}B$	$A^{4n-2} = B^2 = E$ $BA = A^{4n-3}B$

	$1\epsilon_1$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$4\epsilon_4$	$4\epsilon_4$
$\mathscr{R}(D_{2h})$	E	P^2	Q^2	R^2	P^2Q^2	P^2R^2	Q^2R^2	$P^2Q^2R^2$	PQ^2, P^3Q^2 $PQ^2R^2, P^3Q^2R^2$	QR^2, Q^3R P^2QR^2, P^2Q
A_g	1	1	1	1	1	1	1	1	1	1
B_{1g}	1	1	1	1	1	1	1	1	1	−1
B_{2g}	1	1	1	1	1	1	1	1	−1	1
B_{3g}	1	1	1	1	1	1	1	1	−1	−1
A_u	1	1	1	1	1	1	1	1	−1	−1
B_{1u}	1	1	1	1	1	1	1	1	−1	1
B_{2u}	1	1	1	1	1	1	1	1	1	−1
B_{3u}	1	1	1	1	1	1	1	1	1	1
E'_α	2	2	−2	2	−2	2	−2	−2	2	0
E''_α	2	2	−2	2	−2	2	−2	−2	−2	0
E'_β	2	2	2	−2	2	−2	−2	−2	0	2
E''_β	2	2	2	−2	2	−2	−2	−2	0	−2
E'_γ	2	−2	2	2	−2	−2	2	−2	0	0
E''_γ	2	−2	2	2	−2	−2	2	−2	0	0
$G_{\alpha\beta}$ $G^+_{\alpha\beta}$	2	2	−2	−2	−2	−2	2	2	0	0
$G^-_{\alpha\beta}$	2	2	−2	−2	−2	−2	2	2	0	0
$G_{\alpha\gamma}$ $G^+_{\alpha\gamma}$	2	−2	−2	2	2	−2	−2	2	0	0
$G^-_{\alpha\gamma}$	2	−2	−2	2	2	−2	−2	2	0	0
$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	2	−2	2	−2	−2	2	−2	2	0	0
$G^-_{\beta\gamma}$	2	−2	2	−2	−2	2	−2	2	0	0
$E'_{\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	0	0
$E''_{\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	0	0
D_{2h}	E									

Table 3 (cont.)

	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_4$	4ϵ
R^2 Q^3R^2	P^2R, P^2R^3 $P^2Q^2R, P^2Q^2R^3$	P, P^3 PR^2, P^3R^2	Q, Q^3 P^2Q, P^2Q^3	R, R^3 Q^2R, Q^2R^3	PQ P^3Q PQ^3R^2 $P^3Q^3R^2$	PQ^3 P^3Q^3 PQR^2 P^3QR^2	PR PR^3 P^3Q^2R $P^3Q^2R^3$	P^3R P^3R^3 PQ^2R PQ^2R^3	Q Q^3 P^2Q P^2Q
	1	1	1	1	1	1	1	1	
	−1	1	−1	−1	−1	−1	−1	−1	
	−1	−1	1	−1	−1	−1	1	1	−
	1	−1	−1	1	1	1	−1	−1	−
	−1	−1	−1	−1	1	1	1	1	
	1	−1	1	1	−1	−1	−1	−1	
	1	1	−1	1	−1	−1	1	1	−
	−1	1	1	−1	1	1	−1	−1	−
	0	−2	0	0	0	0	0	0	
	0	2	0	0	0	0	0	0	
	0	0	−2	0	0	0	0	0	
	0	0	2	0	0	0	0	0	
	2	0	0	−2	0	0	0	0	
	−2	0	0	2	0	0	0	0	
	0	0	0	0	2i	−2i	0	0	
	0	0	0	0	−2i	2i	0	0	
	0	0	0	0	0	0	2i	−2i	
	0	0	0	0	0	0	−2i	2i	
	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	−
	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	
		A	*B*	*C*	*AB*		*AC*		

	$4\epsilon_4$	$4\epsilon_4$	$4\epsilon_2$	$4\epsilon_2$	64 elements		
					$P^4 = Q^4 = R^4 = E$		
					$QP = P^3Q$		
					$RQ = Q^3R$		
					$PR = R^3P$		
	QR	QR^3	PQR	PQR^3			
	Q^3R	Q^3R^3	P^3QR^3	PQ^3R			
R	P^2QR^3	P^2QR	P^3Q^3R	P^3QR			
R^3	$P^2Q^3R^3$	P^2Q^3R	PQ^3R^3	$P^3Q^3R^3$	α	β	γ
	1	1	1	1	1	1	1
	1	1	1	1			
	−1	−1	1	1			
	−1	−1	1	1			
	1	1	−1	−1			
	1	1	−1	−1			
	−1	−1	−1	−1			
	−1	−1	−1	−1			
	0	0	0	0	−1	1	1
	0	0	0	0			
	0	0	0	0	1	−1	1
	0	0	0	0			
	0	0	0	0	1	1	−1
	0	0	0	0			
	0	0	0	0	−1	−1	1
	0	0	0	0			
i	0	0	0	0	−1	1	−1
i	0	0	0	0			
	2i	−2i	0	0	1	−1	−1
	−2i	2i	0	0			
	0	0	2	−2	−1	−1	−1
	0	0	−2	2			
	BC		ABC				

$A^2 = B^2 = C^2 = E$
$AB = BA$
$AC = CA$
$BC = CB$

	$\mathscr{R}_1(D_{2nh})$	E	P^{2n}	Q^2	R^2	Q^2R^2	$P^{2n}Q^2$	$P^{2n}R^2$	$P^{2n}Q^2R^2$	$P^{2p}Q^2$ $P^{4n-2p}Q^2$
		$1\epsilon_1$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1 \leqslant p \leqslant n-1$ $2\epsilon_{2n}$
	A_{1g}	1	1	1	1	1	1	1	1	1
	A_{2g}	1	1	1	1	1	1	1	1	1
	B_{1g}	1	1	1	1	1	1	1	1	1
	B_{2g}	1	1	1	1	1	1	1	1	1
	A_{1u}	1	1	1	1	1	1	1	1	1
	A_{2u}	1	1	1	1	1	1	1	1	1
	B_{1u}	1	1	1	1	1	1	1	1	1
	B_{2u}	1	1	1	1	1	1	1	1	1
$\leqslant l \leqslant n-1;$	E_{lg}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/n\}$
$\leqslant l \leqslant n-1;$	E_{lu}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/n\}$
$\leqslant l \leqslant 2n;$	$E_{l\alpha}$	2	-2	2	2	2	-2	-2	-2	$2\cos\{(2l-1)\,p\pi/$
	$E_{1\beta}$	2	2	-2	2	-2	-2	2	-2	-2
	$E_{2\beta}$	2	2	-2	2	-2	-2	2	-2	-2
$\leqslant l \leqslant n-1;$	$G_{l\beta}$ $G^+_{l\beta}$	2	2	-2	2	-2	-2	2	-2	$-2\cos\{2lp\pi/n\}$
	$G^-_{l\beta}$	2	2	-2	2	-2	-2	2	-2	$-2\cos\{2lp\pi/n\}$
	$E_{1\gamma}$	2	2	2	-2	-2	2	-2	-2	2
	$E_{2\gamma}$	2	2	2	-2	-2	2	-2	-2	2
ven;	G_γ G^+_γ	2	2	2	-2	-2	2	-2	-2	$2(-1)^p$
	G^-_γ	2	2	2	-2	-2	2	-2	-2	$2(-1)^p$
$\leqslant l \leqslant \frac{1}{2}(n-1);$	$G_{l\gamma}$	4	4	4	-4	-4	4	-4	-4	$4\cos\{2lp\pi/n\}$
$\leqslant l \leqslant n;$	$G_{l\alpha\beta}$ $G^+_{l\alpha\beta}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)\,p\pi$
	$G^-_{l\alpha\beta}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)\,p\pi$
dd;	$G_{\alpha\gamma}$ $G^+_{\alpha\gamma}$	2	-2	2	-2	-2	-2	2	2	$2(-1)^p$
	$G^-_{\alpha\gamma}$	2	-2	2	-2	-2	-2	2	2	$2(-1)^p$
$\leqslant l \leqslant \frac{1}{2}n;$	$G_{l\alpha\gamma}$	4	-4	4	-4	-4	-4	4	4	$4\cos\{(2l-1)\,p\pi/$
	$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	-2
	$G^-_{\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	-2
ven;	$E_{1\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	$2(-1)^{p+1}$
ven;	$E_{2\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	$2(-1)^{p+1}$
$\leqslant l \leqslant \frac{1}{2}(n-1);$	$G_{l\beta\gamma}$	4	4	-4	-4	4	-4	-4	4	$-4\cos\{2lp\pi/n\}$
dd;	$E_{1\alpha\beta\gamma}$	2	-2	-2	-2	2	2	2	-2	$2(-1)^{p+1}$
dd;	$E_{2\alpha\beta\gamma}$	2	-2	-2	-2	2	2	2	-2	$2(-1)^{p+1}$
$\leqslant l \leqslant \frac{1}{2}n;$	$G_{l\alpha\beta\gamma}$	4	-4	-4	-4	4	4	4	-4	$-4\cos\{(2l-1)\,p\pi$
	D_{2nh}	E								

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TABLE 3 (cont.)

1	$1 \leqslant p \leqslant n-1$	$1 \leqslant p \leqslant n-1$	$1 \leqslant p \leqslant n$	$1 \leqslant p \leqslant n-1$	$1 \leqslant p$
	$2\epsilon_{2n}$	$2\epsilon_{2n}$	$4\epsilon_{4n/\mathrm{hcf}(4n,\,2p-1}$	$2\epsilon_{2n/\mathrm{hcf}(2n,\,p)}$	$4\epsilon_{4n/\mathrm{hcf}(4}$
			$P^{4n+1-2p}R^2$		P^{4n+1-2}
			$P^{2p-1}R^2$		$P^{2p-1}Q$
	$P^{4n-2p}R^2$	$P^{4n-2p}R^2$	$P^{4n+1-2p}$	P^{4n-2p}	P^{4n+1-}
	$P^{2p}R^2$	$P^{2p}Q^2R^2$	P^{2p-1}	P^{2p}	P^{2p-1}
	1	1	1	1	1
	1	1	1	1	1
	1	1	-1	1	-1
	1	1	-1	1	-1
	1	1	1	1	1
	1	1	1	1	1
	1	1	-1	1	-1
	1	1	-1	1	-1
}	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\,\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-$
}	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\,\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-$
$\pi/n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$2\cos\{(2l-1)\,(2$
	2	-2	2	2	-2
	2	-2	-2	2	2
}	$2\cos\{2lp\pi/n\}$	$-2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\,\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\{l(2p$
}	$2\cos\{2lp\pi/n\}$	$-2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\,\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\,(l(2p$
	-2	-2	0	2	0
	-2	-2	0	2	0
	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^{p}$	0
	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^{p}$	0
}	$-4\cos\{2lp\pi/n\}$	$-4\cos\{2lp\pi/n\}$	0	$4\cos\{2pl\pi/n\}$	0
$p\pi/n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$-2\cos\{(2l-1)\,p\pi/n\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$-2\cos\{(2l-1)\,($
$p\pi/n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$-2\cos\{(2l-1)\,p\pi/n\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	$2\cos\{(2l-1)\,p\pi/n\}$	$-2\cos\{(2l-1)\,($
	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^{p}$	0
	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^{p}$	0
$\pi/n\}$	$-4\cos\{(2l-1)\,p\pi/n\}$	$-4\cos\{(2l-1)\,p\pi/n\}$	0	$4\cos\{(2l-1)\,p\pi/n\}$	0
	-2	2	0	2	0
	-2	2	0	2	0
	$2(-1)^{p+1}$	$2(-1)^{p}$	0	$2(-1)^{p}$	0
	$2(-1)^{p+1}$	$2(-1)^{p}$	0	$2(-1)^{p}$	0
$n\}$	$-4\cos\{2lp\pi/n\}$	$4\cos\{2lp\pi/n\}$	0	$4\cos\{2lp\pi/n\}$	0
	$2(-1)^{p+1}$	$2(-1)^{p}$	0	$2(-1)^{p}$	0
	$2(-1)^{p+1}$	$2(-1)^{p}$	0	$2(-1)^{p}$	0
$p\pi/n\}$	$-4\cos\{(2l-1)\,p\pi/n\}$	$4\cos\{(2l-1)\,p\pi/n\}$	0	$4\cos\{(2l-1)\,p\pi/n\}$	0
			A^{2p-1}	A^{2p}	

$p \leqslant n$	$1 \leqslant p \leqslant 2n$	$0 \leqslant p \leqslant 2n-1$			
$(4n, 2p-1)$	$4\epsilon_{4n/\text{hcf}(4n,\,2p-1)}$	$4\epsilon_{4n/\text{hcf}(2n-3,\,p)}$	$4n\epsilon_4$ $0 \leqslant q \leqslant 2n-1$	$4n\epsilon_4$ $0 \leqslant q \leqslant 2n-1$	$4n\epsilon_4$ $1 \leqslant q \leqslant 2n$
$^{-2p}Q^2R^2$	$P^{4n+1-2p}Q^2R^3$	$P^{4n-2p}Q^2R^3$			
$^{1}Q^2R^2$	$P^{4n+1-2p}Q^2R$	$P^{4n-2p}Q^2R$			
$^{1-2p}Q^2$	$P^{2p-1}R^3$	$P^{2p}R^3$	$P^{2q}Q^3$	$P^{2q}Q^3R^2$	$P^{2q-1}Q^3R^2$
$^{-1}Q^2$	$P^{2p-1}R$	$P^{2p}R$	$P^{2q}Q$	$P^{2q}QR^2$	$P^{2q-1}Q$
1	1	1	1	1	1
1	1	1	-1	-1	-1
·1	$(-1)^{n-1}$	$(-1)^n$	1	1	-1
·1	$(-1)^{n-1}$	$(-1)^n$	-1	-1	1
1	-1	-1	-1	-1	-1
1	-1	-1	1	1	1
·1	$(-1)^n$	$(-1)^{n-1}$	-1	-1	1
·1	$(-1)^n$	$(-1)^{n-1}$	1	1	-1
$-1)\,\pi/n\}$	$(-1)^l 2\cos\{l(2p-1)\,\pi/n\}$	$(-1)^l\,2\cos\{2lp\pi/n\}$	0	0	0
$-1)\,\pi/n\}$	$(-1)^{l+1}\,2\cos\{l(2p-1)\,\pi/n\}$	$(-1)^{l+1}\,2\cos\{2lp\pi/n\}$	0	0	0
$(2p-1)\,\pi/2n\}$	$(-1)^l 2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	$(-1)^l 2\cos\{(2l-1)\,p\pi/n\}$	0	0	0
·2	0	0	0	0	0
2	0	0	0	0	0
$2p-1)\,\pi/n\}$	$2\mathrm{i}\sin\{l(2p-1)\,\pi/n\}$	$2\mathrm{i}\sin\{2lp\pi/n\}$	0	0	0
$2p-1)\,\pi/n\}$	$-2\mathrm{i}\sin\{l(2p-1)\,\pi/n\}$	$-2\mathrm{i}\sin\{2lp\pi/n\}$	0	0	0
0	0	0	2	-2	0
0	0	0	-2	2	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
$)\,(2p-1)\,\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)\,(2p-1)\,\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)\,p\pi/n\}$	0	0	0
$)\,(2p-1)\,\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)\,(2p-1)\,\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)\,p\pi/n\}$	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	$2\mathrm{i}$
0	0	0	0	0	$-2\mathrm{i}$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
	$A^{2p-1}C$	$A^{2p}C$	$A^{2q}B$ $0 \leqslant q \leqslant n-1$		$A^{2q-1}B$ $1 \leqslant q \leqslant n$

$4n\epsilon_4$ $\leqslant q \leqslant 2n$	$4n\epsilon_4$ $1 \leqslant q \leqslant 2n$	$4n\epsilon_4$ $0 \leqslant q \leqslant n-1$	$4n\epsilon_4$ $0 \leqslant q \leqslant n-1$	$4n\epsilon_2$ $1 \leqslant q \leqslant n$	$4n\epsilon_2$ $1 \leqslant q \leqslant n$	$64n$ elements
$P^{2q-1}Q^3R^2$ $P^{2q-1}Q$	$P^{2q-1}QR^2$ $P^{2q-1}Q^3$	$P^{4q+2}Q^3R$ $P^{4q+2}QR$ $P^{4q}Q^3R^3$ $P^{4q}QR^3$	$P^{4q+2}Q^3R^3$ $P^{4q+2}QR^3$ $P^{4q}Q^3R$ $P^{4q}QR$	$P^{4q-3}Q^3R^3$ $P^{4q-3}QR$ $P^{4q-1}QR^3$ $P^{4q-1}Q^3R$	$P^{4q-3}QR^3$ $P^{4q-3}Q^3R$ $P^{4q-1}Q^3R^3$ $P^{4q-1}QR$	$P^{4n} = Q^4 = R^4 = E$ $QP = P^{4n-1}Q$; $RQ = Q^3R$ $PR = R^3P$
1	1	1	1	1	1	
−1	−1	−1	−1	−1	−1	
−1	−1	1	1	−1	−1	
1	1	−1	−1	1	1	
−1	−1	1	1	1	1	$\alpha = +1; \beta = +1; \gamma = +1$
1	1	−1	−1	−1	−1	
1	1	1	1	−1	−1	
−1	−1	−1	−1	1	1	
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	$\alpha = -1; \beta = +1; \gamma = +1$
0	0	0	0	0	0	
0	0	0	0	0	0	$\alpha = +1; \beta = -1; \gamma = +1$
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	
0	0	2i	−2i	0	0	$\alpha = +1; \beta = +1; \gamma = -1$
0	0	−2i	2i	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	$\alpha = -1; \beta = -1; \gamma = +1$
0	0	0	0	0	0	
0	0	2i	−2i	0	0	
0	0	−2i	2i	0	0	$\alpha = -1; \beta = +1; \gamma = -1$
0	0	0	0	0	0	
2i	−2i	0	0	0	0	
−2i	2i	0	0	0	0	
0	0	0	0	2	−2	$\alpha = +1; \beta = -1; \gamma = -1$
0	0	0	0	−2	2	
0	0	0	0	0	0	
0	0	0	0	2	−2	
0	0	0	0	−2	2	$\alpha = -1; \beta = -1; \gamma = -1$
0	0	0	0	0	0	
$A^{2q-1}B$ $1 \leqslant q \leqslant n$		$A^{4q}BC$ $0 \leqslant q \leqslant \frac{1}{2}(n-1)$	$A^{4q-2}BC$ $1 \leqslant q \leqslant \frac{1}{2}n$	$A^{4q-3}BC$ $1 \leqslant q \leqslant \frac{1}{2}(n+1)$	$A^{4q-1}BC$ $1 \leqslant q \leqslant \frac{1}{2}n$	$A^{2n} = B^2 = C^2 = E$ $BA = A^{2n-1}B$ $CA = AC$; $CB = BC$

										$1 \leqslant p \leqslant 2n-1$	1
		$1\epsilon_1$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$2\epsilon_{4n/\mathrm{hcf}(4n,\, p}$	2
										$P^{8n-2p}Q^2$	
	$\mathscr{R}_2(D_{4nh})$	E	P^{4n}	Q^2	R^2	Q^2R^2	$P^{4n}Q^2$	$P^{4n}R^2$	$P^{4n}Q^2R^2$	$P^{2p}Q^2$	
	A_{1g}	1	1	1	1	1	1	1	1	1	
	A_{2g}	1	1	1	1	1	1	1	1	1	
	B_{1g}	1	1	1	1	1	1	1	1	1	
	B_{2g}	1	1	1	1	1	1	1	1	1	
	A_{1u}	1	1	1	1	1	1	1	1	1	
	A_{2u}	1	1	1	1	1	1	1	1	1	
	B_{1u}	1	1	1	1	1	1	1	1	1	
	B_{2u}	1	1	1	1	1	1	1	1	1	
$1 \leqslant l \leqslant 2n-1$;	E_{lg}	2	2	2	2	2	2	2	2	$2\cos\{lp\pi/n\}$	2
$1 \leqslant l \leqslant 2n-1$;	E_{lg}	2	2	2	2	2	2	2	2	$2\cos\{lp\pi/n\}$	2
$1 \leqslant l \leqslant 2n$; $G_{l\alpha}$	$G_{l\alpha}^{+}$	2	-2	2	2	2	-2	-2	-2	$2\cos\{(2l-1)\,p\pi/2n\}$	2 co
	$G_{l\alpha}^{-}$	2	-2	2	2	2	-2	-2	-2	$2\cos\{(2l-1)\,p\pi/2n\}$	2 co
	$E_{1\beta}$	2	2	-2	2	-2	-2	2	-2	-2	
	$E_{2\beta}$	2	2	-2	2	-2	-2	2	-2	-2	
$\leqslant l \leqslant 2n-1$; $G_{l\beta}$	$G_{l\beta}^{+}$	2	2	-2	2	-2	-2	2	-2	$2(-1)^{p+1}$	
	$G_{l\beta}^{-}$	2	2	-2	2	-2	-2	2	-2	$2(-1)^{p+1}$	
	$E_{1\gamma}$	2	2	2	-2	-2	2	-2	-2	2	
	$E_{2\gamma}$	2	2	2	-2	-2	2	-2	-2	2	
G_{γ}	G_{γ}^{+}	2	2	2	-2	-2	2	-2	-2	$2(-1)^{p}$	
	G_{γ}^{-}	2	2	2	-2	-2	2	-2	-2	$2(-1)^{p}$	
$1 \leqslant l \leqslant n-1$;	$G_{l\gamma}$	4	4	4	-4	-4	4	-4	-4	$4\cos\{(2l-1)\,p\pi/n\}$	-4
$1 \leqslant l \leqslant 2n$; $G_{l\alpha\beta}$	$G_{l\alpha\beta}^{+}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	2 cos{
	$G_{l\alpha\beta}^{-}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)\,(2p-1)\,\pi/2n\}$	2 cos{
$1 \leqslant l \leqslant n$;	$G_{l\alpha\gamma}$	4	-4	4	-4	-4	-4	4	4	$4\cos\{(2l-1)\,p\pi/2n\}$	-4 c
$G_{\beta\gamma}$	$G_{\beta\gamma}^{+}$	2	2	-2	-2	2	-2	-2	2	-2	
	$G_{\beta\gamma}^{-}$	2	2	-2	-2	2	-2	-2	2	-2	
	$E_{1\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	$2(-1)^{p+1}$	
	$E_{2\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	$2(-1)^{p+1}$	
$1 \leqslant l \leqslant n-1$;	$G_{l\beta\gamma}$	4	4	-4	-4	4	-4	-4	4	$-4\cos\{(2l-1)\,p\pi/n\}$	-4 c
$1 \leqslant l \leqslant n$;	$G_{l\alpha\beta\gamma}$	4	-4	-4	-4	4	4	4	-4	$-4\cos\{(2l-1)\,p\pi/2n\}$	-4 c
	D_{4nh}	E									

T

$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,p)}$	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,p)}$	$1 \leqslant p \leqslant 2n$ $4\epsilon_{8n/\mathrm{hcf}(8n,4p-3)}$	$1 \leqslant p \leqslant 2n-1$ $2\epsilon_{4n/\mathrm{hcf}(4n,p)}$	
$P^{8n-2p}R^2$ $P^{2p}R^2$	$P^{8n-2p}Q^2R^2$ $P^{2p}Q^2R^2$	$P^{4n+3-4p}R^2$ $P^{4p-3}R^2$ $P^{4n+3-4p}$ P^{4p-3}	P^{8n-2p} P^{2p}	
1	1	1	1	
1	1	1	1	
1	1	-1	1	
1	1	-1	1	
1	1	1	1	
1	1	1	1	
1	1	-1	1	
1	1	-1	1	
$2\cos\{lp\pi/n\}$	$2\cos\{lp\pi/n\}$	$2\cos\{l(4p-3)\pi/2n\}$	$2\cos\{lp\pi/n\}$	
$2\cos\{lp\pi/n\}$	$2\cos\{lp\pi/n\}$	$2\cos\{l(4p-3)\pi/2n\}$	$2\cos\{lp\pi/n\}$	
$\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	2i si
$\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	−2i
2	-2	2	2	
2	-2	-2	2	
$2(-1)^p$	$2(-1)^{p+1}$	0	$2(-1)^p$	2i si
$2(-1)^p$	$2(-1)^{p+1}$	0	$2(-1)^p$	−2i s
-2	-2	0	2	
-2	-2	0	2	
$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	
$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	
$4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	
$s\{(2l-1)(2p-1)\pi/2n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	2 co
$s\{(2l-1)(2p-1)\pi/2n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos(2l-1)(2p-1)\pi/2n\}$	2 co
$4\cos\{(2l-1)p\pi/2n\}$	$-4\cos\{(2l-1)p\pi/2n\}$	0	$4\cos\{(2l-1)p\pi/2n\}$	
-2	2	0	2	
-2	2	0	2	
$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	
$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	
$4\cos\{(2l-1)p\pi/n\}$	$4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	
$4\cos\{(2l-1)p\pi/2n\}$	$4\cos\{(2l-1)p\pi/2n\}$	0	$4\cos\{(2l-1)p\pi/2n\}$	
		A^{2p-1} $A^{4n+1-2p}$ $1 \leqslant p \leqslant n$	A^{2p}	

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TABLE 3 (*cont.*)

$1 \leqslant p \leqslant 4n$ $4\epsilon_{8n/\mathrm{hcf}(2p-1,\,n)}$ $P^{4n+1-2p}Q^2R^3$ $P^{4n+1-2p}Q^2R$ $P^{2p-1}R^3$ $P^{2p-1}R$	$0 \leqslant p \leqslant 4n-1$ $4\epsilon_{4/\mathrm{hcf}(n,\,p)}$ $P^{8n-2p}Q^2R^3$ $P^{8n-2p}Q^2R$ $P^{2p}R^3$ $P^{2p}R$	$1 \leqslant p \leqslant 2n$ $4\epsilon_{8n/\mathrm{hcf}(8n,\,4p-3)}$ $P^{4n+3-4p}Q^2R^2$ $P^{4p-3}Q^2R^2$ $P^{4n+3-4p}Q^2$ $P^{4p-3}Q^2$	$8n\epsilon_4$ $0 \leqslant q \leqslant 4n-1$ $P^{2q}Q^3$ $P^{2q}Q$	$8n\epsilon_4$ $0 \leqslant q \leqslant 4n-$ $P^{2q}Q^3R^2$ $P^{2q}QR^2$
1	1	1	1	1
1	1	1	-1	-1
-1	$(-1)^p$	-1	1	1
-1	$(-1)^p$	-1	-1	-1
-1	-1	1	-1	-1
-1	-1	1	1	1
1	$(-1)^{p+1}$	-1	-1	-1
1	$(-1)^{p+1}$	-1	1	1
$2\cos\{l(2p-1)\pi/2n\}$	$2\cos\{lp\pi/n\}$	$2\cos\{l(4p-3)\pi/2n\}$	0	0
$-2\cos\{l(2p-1)\pi/2n\}$	$-2\cos\{lp\pi/n\}$	$2\cos\{l(4p-3)\pi/2n\}$	0	0
$i\sin\{(2l+1)(2p-1)\pi/4n\}$	$(-1)^{l+1}2\cos\{(2l-1)\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$	0	0
$2i\sin\{(2l+1)(2p-1)\pi/4n\}$	$(-1)^{l+1}2\cos\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$	0	0
0	0	-2	0	0
0	0	2	0	0
$\sin\{(2l-1)(2p-1)\pi/2n\}$	$2i\sin\{(2l-1)p\pi/n\}$	0	0	0
$i\sin\{(2l-1)(2p-1)\pi/2n\}$	$-2i\sin\{(2l-1)p\pi/n\}$	0	0	0
0	0	0	2	-2
0	0	0	-2	2
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
$\cos\{(2l-1)(2p-1)\pi/4n\}$	$2i\sin\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$	0	0
$\cos\{(2l-1)(2p-1)\pi/4n\}$	$-2i\sin\{(2l-1)p\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
$A^{2p-1}C$ $1 \leqslant p \leqslant 2n$	$A^{2p}C$ $0 \leqslant p \leqslant 2n-1$		$A^{2q}B$ $0 \leqslant q \leqslant 2n-1$	

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	$8n\epsilon_4$	$8n\epsilon_4$	$8n\epsilon_4$	$8n\epsilon_4$	$8n\epsilon_4$	$8n\epsilon_4$	$128n$
-1	$1 \leqslant q \leqslant 4n$	$1 \leqslant q \leqslant 4n$	$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	
			$P^{4q-2}Q^3R^3$	$P^{4q-2}Q^3R$	$P^{4q-1}Q^3R^3$	$P^{4q-1}Q^3R$	$P^{8n} =$
			$P^{4q-4}Q^3R$	$P^{4q-4}Q^3R^3$	$P^{4q-3}Q^3R$	$P^{4q-3}Q^3R^3$	$QP =$
	$P^{2q-1}Q^3R^2$	$P^{2q-1}QR^2$	$P^{4q-2}QR^3$	$P^{4q-2}QR$	$P^{4q-1}QR^3$	$P^{4q-1}QR$	RP
	$P^{2q-1}Q$	$P^{2q-1}Q^3$	$P^{4q-4}QR$	$P^{4q-4}QR^3$	$P^{4q-3}QR$	$P^{4q-3}QR^3$	RQ
	1	1	1	1	1	1	
	-1	-1	-1	-1	-1	-1	
	-1	-1	1	1	-1	-1	
	1	1	-1	-1	1	1	
	-1	-1	1	1	1	1	
	1	1	-1	-1	-1	-1	$\alpha = +1; \beta$
	1	1	1	1	-1	-1	
	-1	-1	-1	-1	1	1	
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	0	0	0	0	0	0	$\alpha = -1; \beta$
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	0	0	0	0	0	0	$\alpha = +1; \beta$
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	0	0	2i	-2i	0	0	$\alpha = +1; \beta$
	0	0	-2i	2i	0	0	
	0	0	0	0	0	0	
	0	0	0	0	0	0	$\alpha = -1; \beta$
	0	0	0	0	0	0	
	0	0	0	0	0	0	$\alpha = -1; \beta$
	2i	-2i	0	0	0	0	
	-2i	2i	0	0	0	0	
	0	0	0	0	2	-2	$\alpha = +1; \beta$
	0	0	0	0	-2	2	
	0	0	0	0	0	0	
	0	0	0	0	0	0	$\alpha = -1; \beta$
	$A^{2q-1}B$		$A^{4q}BC$	$A^{4q+2}BC$	$A^{4q+1}BC$	$A^{4q+3}BC$	$A^{4n} = E$
	$1 \leqslant q \leqslant 2n$		$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	$0 \leqslant q \leqslant n-1$	$BA =$
							$CA = AC$

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128n elements

$$P^{8n} = Q^4 = R^4 = E$$
$$QP = P^{4n-1}Q$$
$$RP = PR^3$$
$$RQ = Q^3R$$

$= +1; \beta = +1; \gamma = +1$

$= -1; \beta = +1; \gamma = +1$

$= +1; \beta = -1; \gamma = +1$

$= +1; \beta = +1; \gamma = -1$

$= -1; \beta = -1; \gamma = +1$

$= -1; \beta = +1; \gamma = -1$

$= +1; \beta = -1; \gamma = -1$

$= -1; \beta = -1; \gamma = -1$

$$A^{4n} = B^2 = C^2 = E$$
$$BA = A^{2n-1}B$$
$$CA = AC; CB = BC$$

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	$\mathscr{R}_2(D_{(4n-2)h})$	$1\epsilon_1$ E	$1\epsilon_2$ P^{4n-2}	$1\epsilon_2$ Q^2	$1\epsilon_2$ R^2	$1\epsilon_2$ Q^2R^2	$1\epsilon_2$ $P^{4n-2}Q^2$	$1\epsilon_2$ $P^{4n-2}R^2$	$1\epsilon_2$ $P^{4n-2}Q^2R^2$	$1 \leqslant p \leqslant 2n-2$ $2\epsilon_{4n-2}$ $P^{8n-4-2p}Q^2$ $P^{2p}Q^2$
	A_{1g}	1	1	1	1	1	1	1	1	1
	A_{2g}	1	1	1	1	1	1	1	1	1
	B_{1g}	1	1	1	1	1	1	1	2	1
	B_{2g}	1	1	1	1	1	1	1	1	1
	A_{1u}	1	1	1	1	1	1	1	1	1
	A_{2u}	1	1	1	1	1	1	1	1	1
	B_{1u}	1	1	1	1	1	1	1	1	1
	B_{2u}	1	1	1	1	1	1	1	1	1
$\leqslant l \leqslant 2n-2;$	E_{lg}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/(2n-1)\}$
$\leqslant l \leqslant 2n-2;$	E_{lu}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/(2n-1)\}$
	$E_{1\alpha}$	2	-2	2	2	2	-2	-2	-2	$2(-1)^p$
	$E_{2\alpha}$	2	-2	2	2	2	-2	-2	-2	$2(-1)^p$
$\leqslant l \leqslant n-1;$	$G_{l\alpha}$	4	-4	4	4	4	-4	-4	-4	$4\cos\{(2l-1)p\pi/(2n-1)\}$
	$E_{1\beta}$	2	2	-2	2	-2	-2	2	-2	-2
	$E_{2\beta}$	2	2	-2	2	-2	-2	2	-2	-2
$\leqslant l \leqslant n-1;$	$G_{l\beta}$	4	4	-4	4	-4	-4	4	-4	$-4\cos\{2lp\pi/(2n-1)\}$
	$E_{1\gamma}$	2	2	2	-2	-2	2	-2	-2	2
	$E_{2\gamma}$	2	2	2	-2	-2	2	-2	-2	2
$\leqslant l \leqslant 2n-2;$	$G_{l\gamma}$ $\{G^+_{l\gamma}$	2	2	2	-2	-2	2	-2	-2	$2\cos\{2lp\pi/(2n-1)\}$
	$G^-_{l\gamma}$	2	2	2	-2	-2	2	-2	-2	$2\cos\{2lp\pi/(2n-1)\}$
$\leqslant l \leqslant 2n-1;$	$G_{l\alpha\beta}$ $\{G^+_{l\alpha\beta}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$
	$G^-_{l\alpha\beta}$	2	-2	-2	2	-2	2	-2	2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$
	$G_{\alpha\gamma}$ $\{G^+_{\alpha\gamma}$	2	-2	2	-2	-2	-2	2	2	$2(-1)^p$
	$G^-_{\alpha\gamma}$	2	-2	2	-2	-2	-2	2	2	$2(-1)^p$
$\leqslant l \leqslant n-1;$	$G_{l\alpha\gamma}$	4	-4	4	-4	-4	-4	4	4	$4\cos\{(2l-1)p\pi/(2n-1)\}$
	$G_{\beta\gamma}$ $\{G^+_{\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	-2
	$G^-_{\beta\gamma}$	2	2	-2	-2	2	-2	-2	2	-2
$\leqslant l \leqslant n-1;$	$G_{l\beta\gamma}$	4	4	-4	-4	4	-4	-4	4	$-4\cos\{2lp\pi/(2n-1)\}$
$\leqslant l \leqslant 4n-2;$	$E_{l\alpha\beta\gamma}$	2	-2	-2	-2	2	2	2	-2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$
	$D_{(4n-2)h}$	E								

$1 \leqslant p \leqslant 2n-2$	$1 \leqslant p \leqslant 2n-2$	$1 \leqslant p \leqslant 2n-1$	$1 \leqslant p \leqslant 2n-2$	
$2\epsilon_{4n-2}$	$2\epsilon_{4n-2}$	$4\epsilon_{(8n-4)/\mathrm{hcf}\,(4n-2,\,2p-1)}$	$2\epsilon_{(4n-2)/\mathrm{hcf}\,(4n-2,\;p)}$	
		$P^{4n-3+2p}Q^2$		
		$P^{4n-1-2p}Q^2$		
$P^{8n-4-2p}R^2$	$P^{8n-4-2p}Q^2R^2$	$P^{8n-3-2p}$	$P^{8n-4-2p}$	
$P^{2p}R^2$	$P^{2p}Q^2R^2$	P^{2p-1}	P^{2p}	
1	1	1	1	
1	1	1	1	
1	1	-1	1	
1	1	-1	1	
1	1	1	1	
1	1	1	1	
1	1	-1	1	
1	1	-1	1	
$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	2
$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	-2
$2(-1)^p$	$2(-1)^p$	0	$2(-1)^p$	
$2(-1)^p$	$2(-1)^p$	0	$2(-1)^p$	
$4\cos\,(2l-1)\,p\pi/(2n-1)\}$	$4\cos\{(2l-1)\,p\pi/(2n-1)\}$	0	$4\cos\{(2l-1)p\pi/(2n-1)\}$	
2	-2	0	2	
2	-2	0	2	
$4\cos\{2lp\pi/(2n-1)\}$	$-4\cos\{2lp\pi/(2n-1)\}$	0	$4\cos\{2lp\pi/(2n-1)\}$	
-2	-2	2	2	
-2	-2	-2	2	
$-2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	2
$-2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$-2i$
$2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$-2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	$2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$2i$
$2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$-2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	$2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$-2i$
$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	
$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	
$-4\cos\{(2l-1)\,p\pi/(2n-1)\}$	$-4\cos\{(2l-1)\,p\pi/(2n-1)\}$	0	$4\cos\{(2l-1)\,p\pi/(2n-1)\}$	
-2	2	0	2	
-2	2	0	2	
$-4\cos\{2lp\pi/(2n-1)\}$	$4\cos\{2lp\pi/(2n-1)\}$	0	$4\cos\{2lp\pi/(2n-1)\}$	
$-2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$2\cos\{(2l-1)\,p\pi/(2n-1)\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	$2\cos\{(2l-1)\,p\pi(2n-1)\}$	2
		A^{2p-1}	A^{2p}	

Table 3 (*cont.*)

$1 \leqslant p \leqslant 4n-2$ $4\epsilon_{(4n-2)/\mathrm{hcf}\,(4n-2,\,2p-1)}$ $P^{4n-1-2p}Q^3R^3$ $P^{4n-3+2p}Q^3R$ $P^{8n-3-2p}QR^3$ $P^{2p-1}QR$	$0 \leqslant p \leqslant 4n-3$ $4\epsilon_{(8n-4)/\mathrm{hcf}\,(2n-1,\,p)}$ $P^{4n-2-2p}Q^3R^3$ $P^{4n-2+2p}Q^3R$ $P^{8n-4-2p}QR^3$ $P^{2p}QR$	$1 \leqslant p \leqslant 2n-1$ $4\epsilon_{(8n-4)/\mathrm{hcf}\,/(4n-2,\,2p-1)}$ $P^{4n-3+2p}Q^2R^2$ $P^{4n-1-2p}Q^2R^2$ $P^{8n-3-2p}R^2$ $P^{2p-1}R^2$	$(8n-4)\,\epsilon_4$ $0 \leqslant q \leqslant 4n-3$ $P^{2q}R^3$ $P^{2q}R$
1	1	1	1
1	1	1	−1
1	−1	−1	−1
1	−1	−1	1
−1	−1	1	1
−1	−1	1	−1
−1	1	−1	−1
−1	1	−1	1
$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	0
$2\cos\{l(2p-1)\,\pi/(n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\,\pi/(2n-1)\}$	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	2
0	0	0	−2
0	0	0	0
0	0	−2	0
0	0	2	0
$2\mathrm{i}\sin\{l(2p-1)\,\pi/(2n-1)\}$	$2\mathrm{i}\sin\{2lp\pi/(2n-1)\}$	$-2\cos\{l(2p-1)\,\pi/(2n-1)\}$	0
$2\mathrm{i}\sin\{l(2p-1)\,\pi/(2n-1)\}$	$-2\mathrm{i}\sin\{2lp\pi/(2n-1)\}$	$-2\cos\{l(2p-1)\,\pi/(2n-1)\}$	0
$2\mathrm{i}\sin\{(2l-1)\,(2p-1)\,\pi/(2n-1)\}$	$2\mathrm{i}\sin\{(2l-1)\,(2p+1)\,\pi/(4n-2)\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	0
$2\mathrm{i}\sin\{(2l-1)\,(2p-1)\,\pi/(2n-1)\}$	$-2\mathrm{i}\sin\{(2l-1)\,(2p+1)\,\pi/(4n-2)\}$	$2\cos\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
$2\sin\{(2l-1)\,(2p-1)\,\pi/(4n-2)\}$	$2\sin\{(2l-1)\,p\pi/(2n-1)\}$	$-2\cos\{(2l-1)\,(2p-1)\pi/(4n-2)\}$	0
$A^{2p-1}BC$ $1 \leqslant p \leqslant 2n-1$	$A^{2p}BC$ $0 \leqslant p \leqslant 2n-2$		$A^{2q}C$ $0 \leqslant q \leqslant n$

$(8n-4)\,\epsilon_4$ $0 \leqslant q \leqslant 4n-3$ $P^2Q^2R^3$ $P^{2q}Q^2R$	$(8n-4)\epsilon_4$ $1 \leqslant q \leqslant 4n-2$ $P^{8n-2q+1}Q^2R^3$ $P^{8n-2q+1}R$	$(8n-4)\,\epsilon_4$ $1 \leqslant q \leqslant 4n-2$ $P^{8n-2q+1}Q^2R$ $P^{8n-2q+1}R^3$	$(8n-4)\,\epsilon_4$ $0 \leqslant q \leqslant 2n-2$ $P^{4n-4q}Q^3R^2$ $P^{4n-4q}QR^2$ $P^{4n-4q}Q^3$ $P^{4n-4q}Q$	$(8n-4)\,\epsilon_4$ $0 \leqslant q \leqslant 2n-2$ $P^{4n-4q+2}Q^3R^2$ $P^{4n-4q+2}QR^2$ $P^{4n-4q+2}Q^3$ $P^{4n-4q+2}Q$	$(8n-4)\,\epsilon_4$ $1 \leqslant q \leqslant 2n-1$ $P^{8n-1-4q}Q^3R^2$ $P^{8n-1-4q}QR^2$ $P^{8n-3-4q}Q^3$ $P^{8n-3-4q}Q$	$(8n-4)\,\epsilon_4$ $1 \leqslant q \leqslant 2n-1$ $P^{8n-3-4q}Q^3R^2$ $P^{8n-3-4q}QR^2$ $P^{8n-1-4q}Q^3$ $P^{8n-1-4q}Q$
1	1	1	1	1	1	1
−1	−1	−1	−1	−1	−1	−1
−1	1	1	−1	−1	1	1
1	−1	−1	1	1	−1	−1
1	1	1	−1	−1	−1	−1
−1	−1	−1	1	1	1	1
−1	1	1	1	1	−1	−1
1	−1	−1	−1	−1	1	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	2	−2	0	0
0	0	0	−2	2	0	0
0	0	0	0	0	0	0
−2	0	0	0	0	0	0
2	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	2i	−2i
0	0	0	0	0	−2i	2i
0	0	0	0	0	0	0
0	2i	−2i	0	0	0	0
0	−2i	2i	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
	$A^{4n-2q-1}C$ $1 \leqslant q \leqslant 2n-1$		$A^{4n-4-4q}B$ $0 \leqslant q \leqslant n-1$	$A^{4n-4q-2}B$ $1 \leqslant q \leqslant n-1$	$A^{4n-4q+1}B$ $1 \leqslant q \leqslant n$	$A^{4n-4q-1}B$ $1 \leqslant q \leqslant n-1$

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4) ϵ_4	$(128n-64)$ elements
$2n-1$	
$^{-4q}Q^3R^2$	$P^{8n-4}=Q^4=R^4=E$
$^{-4q}QR^2$	$QP=P^{4n-3}Q^3$
$^{1-4q}Q^3$	$RP=P^{8n-5}R$
^{1-4q}Q	$RQ=QR^3$
1	
1	
1	
1	
1	$\alpha=+1;\beta=+1;\gamma=+1$
1	
1	
1	
)	
)	
)	
)	$\alpha=-1;\beta=+1;\gamma=+1$
)	
)	
)	$\alpha=+1;\beta=-1;\gamma=+1$
)	
)	
)	$\alpha=+1;\beta=+1;\gamma=-1$
)	
)	
)	$\alpha=-1;\beta=-1;\gamma=+1$
)	
2i	
2i	$\alpha=-1;\beta=+1;\gamma=-1$
)	
)	
)	$\alpha=+1;\beta=-1;\gamma=-1$
)	
)	$\alpha=-1;\beta=-1;\gamma=-1$
^{q-1}B	$A^{4n-2}=B^2=C^2=E$
$n-1$	$BA=A^{4n-3}B$
	$CA=AC$
	$CB=BC$

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	$1\epsilon_1$	$1\epsilon_2$	$6\epsilon_4$	$4\epsilon_6$	$4\epsilon_6$	$4\epsilon_3$	$4\epsilon_3$	24 elements
$\mathscr{R}(T)$	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q	PR QR PQR P^2R	P^3QR^2 P^3R^2 P^2QR^2 P^2R^2	R^2 PR^2 QR^2 PQR^2	R P^2QR P^3QR P^3R	$P^4 = Q^4 = R^3 = E$ $P^2 = Q^2$ $QP = P^3Q$; $RP = QR$ $RQ = PQR$
A	1	1	1	1	1	1	1	$\alpha = +1$
$E\begin{cases}E^+\end{cases}$	1	1	1	ω	ω^*	ω^*	ω	
E^-	1	1	1	ω^*	ω	ω	ω^*	
T	3	3	-1	0	0	0	0	
$E_{\frac{1}{2}}$	2	-2	0	1	1	-1	-1	$\alpha = -1$
$G_{\frac{3}{2}}\begin{cases}G_{\frac{3}{2}}^+\end{cases}$	2	-2	0	ω	ω^*	$-\omega^*$	$-\omega$	
$G_{\frac{3}{2}}^-$	2	-2	0	ω^*	ω	$-\omega$	$-\omega^*$	
T	E		A B AB	AC BC ABC		C^2 AC^2 BC^2 ABC^2	C	$A^2 = B^2 = C^3 = E$ $AB = BA$ $CA = BC$ $CB = ABC$

$(\omega = e^{\frac{2}{3}\pi i})$

$\mathscr{R}_1(T_h) = \mathscr{R}(T) \times \{E, T\}$

	$1\epsilon_1$	$1\epsilon_2$	$6\epsilon_4$	$4\epsilon_3$	$4\epsilon_3$	$4\epsilon_6$	$4\epsilon_6$	$1\epsilon_4$	$1\epsilon_4$	$6\epsilon_2$	$4\epsilon_{12}$	$4\epsilon_{12}$	$4\epsilon_{12}$	$4\epsilon_{12}$	48 elements
$\mathscr{R}_2(T_h)$	E	P^2	PQ, P^3Q Q, P^2Q P, P^3	P^2QR P^3QR P^3R R	PR^2 QR^2 PQR^2 R^2	PQR P^2R QR PR	P^2QR^2 P^2R^2 P^3R^2 P^3QR^2	T	P^2T	P^3QT P^2QT P^3T PQT QT PT	P^2QRT P^3QRT P^3RT RT	P^3R^2T P^2QR^2T P^3QR^2T P^2R^2T	PR^2T QR^2T PQR^2T R^2T	QRT $PQRT$ PRT P^2RT	$P^4 = Q^4 = R^3 = T^4 = E$ $P^2 = Q^2 = T^2$ $QP = P^3Q$; $RP = QR$ $RQ = PQR$; $TR = RT$ $TP = PT$; $TQ = QT$
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\alpha = +1$
$E_g\begin{cases}E_g^+\end{cases}$	1	1	1	ω	ω^*	ω	ω^*	1	1	1	ω	ω^*	ω^*	ω	
E_g^-	1	1	1	ω^*	ω	ω^*	ω	1	1	1	ω^*	ω	ω	ω^*	
T_g	3	3	-1	0	0	0	0	3	3	-1	0	0	0	0	
A_u	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	
$E_u\begin{cases}E_u^+\end{cases}$	1	1	1	ω	ω^*	ω	ω^*	-1	-1	-1	$-\omega$	$-\omega^*$	$-\omega^*$	$-\omega$	
E_u^-	1	1	1	ω^*	ω	ω^*	ω	-1	-1	-1	$-\omega^*$	$-\omega$	$-\omega$	$-\omega^*$	
T_u	3	3	-1	0	0	0	0	-3	-3	1	0	0	0	0	
$G_\alpha\begin{cases}G_\alpha^+\end{cases}$	2	-2	0	-1	-1	1	1	$-2i$	$2i$	0	i	$-i$	i	$-i$	$\alpha = -1$
G_α^-	2	-2	0	-1	-1	1	1	$2i$	$-2i$	0	$-i$	i	$-i$	i	
$G'_\alpha\begin{cases}G_\alpha'^+\end{cases}$	2	-2	0	$-\omega$	$-\omega^*$	ω	ω^*	$-2i$	$2i$	0	$i\omega$	$-i\omega^*$	$i\omega^*$	$-i\omega$	
$G_\alpha'^-$	2	-2	0	$-\omega^*$	$-\omega$	ω^*	ω	$2i$	$-2i$	0	$-i\omega^*$	$i\omega$	$-i\omega$	$i\omega^*$	
$G''_\alpha\begin{cases}G_\alpha'^+\end{cases}$	2	-2	0	$-\omega^*$	$-\omega$	ω^*	ω	$-2i$	$2i$	0	$i\omega^*$	$-i\omega$	$i\omega$	$-i\omega^*$	
$G_\alpha'^-$	2	-2	0	$-\omega$	$-\omega^*$	ω	ω^*	$2i$	$-2i$	0	$-i\omega$	$i\omega^*$	$-i\omega^*$	$i\omega$	
T_h	E		A B AB	C	C^2 AC^2 BC^2 ABC^2	AC BC ABC		I		AI BI ABI	CI		C^2I AC^2I BC^2I ABC^2I	ACI BCI $ABCI$	$A^2 = B^2 = C^3 = I^2 = E$ $BA = AB$; $CA = BC$ $CB = ABC$; $IA = AI$ $IB = BI$; $IC = CI$

$(\omega = e^{\frac{2}{3}\pi i})$

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TABLE 3 (*cont.*)

	$1\epsilon_1$	$1\epsilon_2$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_3$	$12\epsilon_4$	$6\epsilon_8$	$6\epsilon_8$	48 elements
$\mathscr{R}_1(O)$	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q	PR^2, P^2QR QR^2, PQR P^2R, P^2R^2 P^3R, P^3QR^2	R^2, R PR, PQR^2 QR, P^3R^2 P^3QR, P^2QR^2	S, P^2S PQS, P^3QS R^2S, P^2R^2S RS, P^2RS PR^2S, P^3R^2S QRS, P^2QRS	PS, P^2QS QR^2S, PQR^2S P^2QS, P^3QRS	QS, P^3S $PRS, PQRS$ P^2QR^2S, P^3QR^2S	$P^4 = Q^4 = R^3 = S^4 = E$ $P^2 = Q^2 = S^2$ $QP = P^3Q$; $RP = QR$ $RQ = P^3QR$; $SP = P^2QS$ $SQ = P^3S$; $SR = R^2S$
A_1	1	1	1	1	1	1	1	1	} $\alpha = +1$
A_2	1	1	1	1	1	−1	−1	−1	
E	2	2	2	−1	−1	0	0	0	
T_1	3	3	−1	0	0	−1	1	1	
T_2	3	3	−1	0	0	1	−1	−1	
$E_{\frac{1}{2}}$	2	−2	0	1	−1	0	$\sqrt{2}$	$-\sqrt{2}$	} $\alpha = -1$
$E_{\frac{5}{2}}$	2	−2	0	1	−1	0	$-\sqrt{2}$	$\sqrt{2}$	
$G_{\frac{3}{2}}$	4	−4	0	−1	1	0	0	0	
O	E		A B AB	AC^2 BC^2, ABC	C^2, C AC, ABC^2 BC	D ABD C^2D CD AC^2D BCD	AD BC^2D, ABC^2D	BD $ACD, ABCD$	$A^2 = B^2 = C^3 = D^2 = E$ $BA = AB$; $CA = BC$ $CB = ABC$; $DA = BD$ $DB = AD$; $DC = C^2D$

	$1\epsilon_1$	$1\epsilon_2$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_3$	$12\epsilon_2$	$6\epsilon_4$	$6\epsilon_4$	48 elements
$\mathscr{R}_2(O)$	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q	PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^2R, P^2R^2	R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	S; P^2S PQS; P^3QS R^2S; RS PR^2S; P^2QRS QRS; P^3R^2S P^2RS; P^2R^2S	PS P^2QS $PQRS$ P^3QR^2S QR^2S P^3RS	QS P^3S PRS P^2QR^2S PQR^2S P^3QRS	$P^4 = Q^4 = R^3 = S^2 = E$ $P^2 = Q^2$ $QP = P^3Q$; $RP = QR$ $RQ = PQR$; $SP = P^2QS$ $SQ = P^3S$; $SR = R^2S$
A_1	1	1	1	1	1	1	1	1	} $\alpha = +1$
A_2	1	1	1	1	1	−1	−1	−1	
E	2	2	2	−1	−1	0	0	0	
T_1	3	3	−1	0	0	−1	1	1	
T_2	3	3	−1	0	0	1	−1	−1	
$G_{\frac{3}{2}}$	4	−4	0	−1	1	0	0	0	} $\alpha = -1$
$G_{\frac{1}{2}}$ { $G_{\frac{1}{2}}^{+}$	2	−2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	
$G_{\frac{1}{2}}$ { $G_{\frac{1}{2}}^{-}$	2	−2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	
O	E		A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D; CD AC^2D BCD	AD $ABCD$ BC^2D	BD ACD ABC^2D	$A^2 = B^2 = C^3 = D^2 = E$ $BA = AB$; $CA = BC$ $CB = ABC$; $DA = BD$ $DB = AD$; $DC = C^2D$

$\mathscr{E}_1(O_h)$	$1\epsilon_1$ E	$1\epsilon_2$ P^2	$1\epsilon_2$ S^2	$1\epsilon_2$ P^2S^2	$6\epsilon_4$ P, P^3 Q, P^2Q PQ, P^3Q	$8\epsilon_6$ PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^2R, P^2R^2	$8\epsilon_3$ R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	$24\epsilon_4$ S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^2RS, P^2RS^3 P^2S, P^2S^3 P^3QS, P^3QS^3 $P^2R^2S, P^2R^2S^3$ $P^3R^2S, P^3R^2S^3$ P^2QRS, P^2QRS^3 RS, RS^3	$12\epsilon_8$ PS, QS^3 P^2QS, P^3S^3 $PQRS, PRS^3$ P^3RS, P^3QRS^3 QR^2S, PQR^2S^3 $P^3QR^2S, P^2QR^2S^3$	P PI P³(PQ P²QI
A_{1g}	1	1	1	1	1	1	1	1	1	
A_{2g}	1	1	1	1	1	1	1	−1	−1	
E_g	2	2	2	2	2	−1	−1	0	0	
T_{1g}	3	3	3	3	−1	0	0	−1	1	
T_{2g}	3	3	3	3	−1	0	0	1	−1	
A_{1u}	1	1	1	1	1	1	1	1	1	
A_{2u}	1	1	1	1	1	1	1	−1	−1	
E_u	2	2	2	2	2	−1	−1	0	0	
T_{1u}	3	3	3	3	−1	0	0	−1	1	
T_{2u}	3	3	3	3	−1	0	0	1	−1	
G_α	4	−4	4	−4	0	2	−2	0	0	
K_α^+	4	−4	4	−4	0	−1	1	0	0	
K_α^-	4	−4	4	−4	0	−1	1	0	0	
E_β	2	2	−2	−2	2	2	2	0	0	
G_β^+	2	2	−2	−2	2	−1	−1	0	0	
G_β^-	2	2	−2	−2	2	−1	−1	0	0	
I_β	6	6	−6	−6	−2	0	0	0	0	
$E'_{\alpha\beta g}$	2	−2	−2	2	0	1	−1	0	$\sqrt{2}$	-
$E'_{\alpha\beta u}$	2	−2	−2	2	0	1	−1	0	$-\sqrt{2}$	
$E''_{\alpha\beta g}$	2	−2	−2	2	0	1	−1	0	$-\sqrt{2}$	
$E''_{\alpha\beta u}$	2	−2	−2	2	0	1	−1	0	$\sqrt{2}$	-
$G_{\alpha\beta g}$	4	−4	−4	4	0	−1	1	0	0	
$G_{\alpha\beta u}$	4	−4	−4	4	0	−1	1	0	0	
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	

Table 3 (*cont.*)

$12\epsilon_8$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_6$	$2\epsilon_2$	$2\epsilon_2$	$12\epsilon_4$	$8\epsilon_6$
							PI
							QI
QS, PS^3						PT, P^3T	PQ
P^3S, P^2QS^3						QT, P^2QT	P^2
$PRS, PQRS^3$		$PRS^2, P^3QR^2S^2$	R^2S^2, RS^2			PQT, P^3QT	R^2S
$^3QRS, P^3RS^3$	PS^2, P^3S^2	$QRS^2, P^3R^2S^2$	PR^2S^2, P^2QRS^2			PS^2T, P^3S^2T	PR^2
QR^2S, QR^2S^3	QS^2, P^2QS^2	$PQRS^2, P^2QR^2S^2$	QR^2S^2, P^3QRS^2	T	P^2T	QS^2T, P^2QS^2T	QR^2
$QR^2S, P^3QR^2S^3$	PQS^2, P^3QS^2	$P^2RS^2, P^2R^2S^2$	PQR^2S^2, P^3RS^2	P^2S^2T	S^2T	PQS^2T, P^3QS^2T	PQR
1	1	1	1	1	1	1	1
−1	1	1	1	1	1	1	1
0	2	−1	−1	2	2	2	−1
1	−1	0	0	3	3	−1	0
−1	−1	0	0	3	3	−1	0
1	1	1	1	−1	−1	−1	−1
−1	1	1	1	−1	−1	−1	−1
0	2	−1	−1	−2	−2	−2	1
1	−1	0	0	−3	−3	1	0
−1	−1	0	0	−3	−3	1	0
0	0	2	−2	0	0	0	0
0	0	−1	1	0	0	0	$-i\sqrt{3}$
0	0	−1	1	0	0	0	$i\sqrt{3}$
0	−2	−2	−2	0	0	0	0
0	−2	1	1	0	0	0	$-i\sqrt{3}$
0	−2	1	1	0	0	0	$i\sqrt{3}$
0	2	0	0	0	0	0	0
$-\sqrt{2}$	0	−1	1	2	−2	0	1
$\sqrt{2}$	0	−1	1	−2	2	0	−1
$\sqrt{2}$	0	−1	1	2	−2	0	1
$-\sqrt{2}$	0	−1	1	−2	2	0	−1
0	0	1	−1	4	−4	0	−1
0	0	1	−1	−4	4	0	1
BD				I		AI	AC
						BI	BC
ACD						ABI	AE
ABC^2D							

$8\epsilon_6$	$8\epsilon_6$	$8\epsilon_6$	$8\epsilon_6$	$24\epsilon_4$	$12\epsilon_8$	$12\epsilon_8$
PRT QRT QRT ^2RT $^2S^2T$ R^2S^2T R^2S^2T R^2S^2T	P^3QR^2T P^3R^2T P^2QR^2T P^2R^2T RS^2T P^2QRS^2T P^3QRS^2T P^3RS^2T	R^2T PR^2T QR^2T PQR^2T PRS^2T QRS^2T $PQRS^2T$ P^2RS^2T	RT P^2QRT P^3QRT P^3RT $P^3QR^2S^2T$ $P^3R^2S^2T$ $P^2QR^2S^2T$ $P^2R^2S^2T$	ST, P^2ST $PQST, P^3QST$ R^2ST, P^2R^2ST PR^2ST, P^3R^2ST $QRST, P^2QRST$ P^2RST, RST S^3T, P^2S^3T PQS^3T, P^3QS^3T $R^2S^3T, P^2R^2S^3T$ $PR^2S^3T, P^3R^2S^3T$ QRS^3T, P^2QRS^3T P^2RS^3T, RS^3T	PST, P^2QST $PQRST, P^3RST$ QR^2ST, P^3QR^2ST QS^3T, P^3S^3T PRS^3T, P^3QRS^3T PQR^2S^3T, P^2QRS^3T	QST, P^3ST $PRST, P^3QRS$ PQR^2ST, P^2QR^2S PS^3T, P^2QS^3T $PQRS^3T, P^3RS^3$ $QR^2S^3T, P^3QR^2S^3$
1	1	1	1	1	1	1
1	1	1	1	−1	−1	−1
−1	−1	−1	−1	0	0	0
0	0	0	0	−1	1	1
0	0	0	0	1	−1	−1
−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	1	1	1
1	1	1	1	0	0	0
0	0	0	0	1	−1	−1
0	0	0	0	−1	1	1
0	0	0	0	0	0	0
$\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0
$\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0
0	0	0	0	0	0	0
$\sqrt{3}$	$i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0
$\sqrt{3}$	$-i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0
0	0	0	0	0	0	0
1	1	−1	−1	0	$\sqrt{2}$	$-\sqrt{2}$
−1	−1	1	1	0	$\sqrt{2}$	$-\sqrt{2}$
1	1	−1	−1	0	$-\sqrt{2}$	$\sqrt{2}$
−1	−1	1	1	0	$-\sqrt{2}$	$\sqrt{2}$
−1	−1	1	1	0	0	0
1	1	−1	−1	0	0	0
ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI

$12\epsilon_8$	192 elements
T, P^3ST	$P^4 = Q^4 = R^3 = S^4 = T^2 = E$
$, P^3QRST$	$P^2 = Q^2$
T, P^2QR^2ST	$QP = P^3Q$; $RP = QR$; $RQ = PQR$;
$, P^2QS^3T$	$SP = P^2QS$; $SQ = P^3S$; $SR = R^2S$;
$^3T, P^3RS^3T$	$TP = PT$; $TQ = QT$; $TR = RT$;
$, P^3QR^2S^3T$	$TS = P^2S^3T$
1	
-1	
0	
1	
-1	
-1	$\alpha = +1$; $\beta = +1$
1	
0	
-1	
1	
0	
0	$\alpha = -1$; $\beta = +1$
0	
0	
0	$\alpha = +1$; $\beta = -1$
0	
0	
$\sqrt{2}$	
$\sqrt{2}$	
$\sqrt{2}$	$\alpha = -1$; $\beta = -1$
$\sqrt{2}$	
0	
0	
BDI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$
$ACDI$	$BA = AB$; $CA = BC$; $CB = ABC$
BC^2DI	$DA = BD$; $DB = AD$; $DC = C^2D$;
	$IA = AI$; $IB = BI$; $IC = CI$;
	$ID = DI$.

$\mathscr{R}_2(O_h)$	$1\epsilon_1$ E	$1\epsilon_2$ P^2	$1\epsilon_2$ T^2	$1\epsilon_2$ P^2T^2	$6\epsilon_4$ P, P^3 Q, P^2Q PQ, P^3Q	$8\epsilon_6$ PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^2R, P^2R^2	$8\epsilon_3$ R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	$24\epsilon_2$ $S; ST^2$ $PQS; PQST^2$ $R^2S; R^2ST^2$ $PR^2S; PR^2ST^2$ $QRS; QRST^2$ $P^2RS; P^2RST^2$ $P^2S; P^2ST^2$ $P^3QS; P^3QST^2$ $P^2R^2S; P^2R^2ST^2$ $P^3R^2S; P^3R^2ST^2$ $P^2QRS; P^2QRST^2$ $RS; RST^2$	$12\epsilon_8$ PS P^2QS $PQRS$ P^3RS QR^2S P^3QR^2S PST^2 P^2QST^2 $PQRST^2$ P^3RST^2 QR^2ST^2 $P^3QR^2ST^2$	12ϵ… QS… P^3S… PR… P^3Q… PQR… P^2Q… QST… P^3ST… $PRST$… P^3QR… PQR^2S… P^2QRS…
A_{1g}	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	1	1	1	−1	−1	−1
E_g	2	2	2	2	2	−1	−1	0	0	0
T_{1g}	3	3	3	3	−1	0	0	−1	1	1
T_{2g}	3	3	3	3	−1	0	0	1	−1	−1
A_{1u}	1	1	1	1	1	1	1	1	1	1
A_{2u}	1	1	1	1	1	1	1	−1	−1	−1
E_u	2	2	2	2	2	−1	−1	0	0	0
T_{1u}	3	3	3	3	−1	0	0	−1	1	1
T_{2u}	3	3	3	3	−1	0	0	1	−1	−1
$G'_{\alpha g}$	4	−4	4	−4	0	−1	1	0	0	0
$G'_{\alpha u}$	4	−4	4	−4	0	−1	1	0	0	0
$G''_{\alpha g}$ { $G''^{+}_{\alpha g}$	2	−2	2	−2	0	1	−1	0	$\mathrm{i}\sqrt{2}$	$-\mathrm{i}\sqrt{2}$
$G''^{-}_{\alpha g}$	2	−2	2	−2	0	1	−1	0	$-\mathrm{i}\sqrt{2}$	$\mathrm{i}\sqrt{2}$
$G''_{\alpha u}$ { $G''^{+}_{\alpha u}$	2	−2	2	−2	0	1	−1	0	$\mathrm{i}\sqrt{2}$	$-\mathrm{i}\sqrt{2}$
$G''^{-}_{\alpha u}$	2	−2	2	−2	0	1	−1	0	$-\mathrm{i}\sqrt{2}$	$\mathrm{i}\sqrt{2}$
$E_{1\beta}$	2	2	−2	−2	2	2	2	0	0	0
$E_{2\beta}$	2	2	−2	−2	2	−1	−1	0	0	0
$E_{3\beta}$	2	2	−2	−2	2	−1	−1	0	0	0
I_β	6	6	−6	−6	−2	0	0	0	0	0
$G_{1\alpha\beta}$	4	−4	−4	4	0	2	−2	0	0	0
$G_{2\alpha\beta}$	4	−4	−4	4	0	−1	1	0	0	0
$G_{3\alpha\beta}$	4	−4	−4	4	0	−1	1	0	0	0
O_h	E				A B AB	AC AC^2 ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	BD… ACD… ABC^2…

TABLE 3 (*cont.*)

$2\epsilon_8$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_6$	$2\epsilon_4$	$2\epsilon_4$	$12\epsilon_4$	8
QS P^3S RS QRS R^2S QRS ST^2 ST^2 ST^2 RST^2 R^2ST^2 RST^2	PT^2, P^3T^2 QT^2, P^2QT^2 PQT^2, P^3QT^2	$PRT^2, P^3QR^2T^2$ $QRT^2, P^3R^2T^2$ $PQRT^2, P^2QR^2T^2$ $P^2RT^2, P^2R^2T^2$	R^2T^2, RT^2 PR^2T^2, P^2QRT^2 QR^2T^2, P^3QRT^2 PQR^2T^2, P^3RT^2	T, T^3	P^2T, P^2T^3	PT, P^3T^3 QT, P^2QT^3 PQT, P^3QT^3 PT^3, P^3T QT^3, P^2QT PQT^3, P^3QT	PRT, P^3 QRT, P $PQRT, P$ P^2RT, P
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
0	2	−1	−1	2	2	2	−1
1	−1	0	0	3	3	−1	0
1	−1	0	0	3	3	−1	0
1	1	1	1	−1	−1	−1	−1
1	1	1	1	−1	−1	−1	−1
0	2	−1	−1	−2	−2	−2	1
1	−1	0	0	−3	−3	1	0
1	−1	0	0	−3	−3	1	0
0	0	−1	1	4	−4	0	−1
0	0	−1	1	−4	4	0	1
/2	0	1	−1	2	−2	0	1
/2	0	1	−1	2	−2	0	1
/2	0	1	−1	−2	2	0	−1
/2	0	1	−1	−2	2	0	−1
0	2	−2	−2	0	0	0	0
0	2	1	1	0	0	0	$-\sqrt{3}$
0	2	1	1	0	0	0	$\sqrt{3}$
0	−2	0	0	0	0	0	0
0	0	−2	2	0	0	0	0
0	0	1	−1	0	0	0	$-\sqrt{3}$
0	0	1	−1	0	0	0	$\sqrt{3}$
BD CD C^2D				I		AI BI ABI	AC BC AB

8ϵ_{12}	8ϵ_{12}	8ϵ_{12}	8ϵ_{12}	24ϵ_2	12ϵ_8	12ϵ_8
				ST; ST^3 $PQST$; $PQST^3$ R^2ST; R^2ST^3 PR^2ST; PR^2ST^3 $QRST$; $QRST^3$ P^2RST; P^2RST^3 P^2ST; P^2ST^3 P^3QST; P^3QST^3	PST P^2QST $PQRST$ P^3RST QR^2ST P^3QR^2ST PST^3 P^2QST^3	QST P^3ST $PRST$ P^3QRST PQR^2ST P^2QRST QST^3 P^3ST^3
$P^3QR^2T^3$, $P^3R^2T^3$, $P^2QR^2T^3$, $P^2R^2T^3$	PRT^3, P^3QR^2T QRT^3, P^3R^2T $PQRT^3$, P^2QR^2T P^2RT^3, P^2R^2T	R^2T, RT^3 PR^2T, P^2QRT^3 QR^2T, P^3QRT^3 PQR^2T, P^3RT^3	R^2T^3, RT PR^2T^3, P^2QRT QR^2T^3, P^3QRT PQR^2T^3, P^3RT	P^2R^2ST; $P^2R^2ST^3$ P^3R^2ST; $P^3R^2ST^3$ P^2QRST; P^2QRST^3 RST; RST^3	$PQRST^3$ P^3RST^3 QR^2ST^3 $P^3QR^2ST^3$	$PRST^3$ P^3QRST^3 PQR^2ST^3 P^2QRST^3
1	1	1	1	1	1	1
1	1	1	1	−1	−1	−1
-1	−1	−1	−1	0	0	0
0	0	0	0	−1	1	1
0	0	0	0	1	−1	−1
-1	−1	−1	−1	−1	−1	−1
-1	−1	−1	−1	1	1	1
1	1	1	1	0	0	0
0	0	0	0	1	−1	−1
0	0	0	0	−1	1	1
-1	−1	1	1	0	0	0
1	1	−1	−1	0	0	0
1	1	−1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$
1	1	−1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$
-1	−1	1	1	0	$-i\sqrt{2}$	$i\sqrt{2}$
-1	−1	1	1	0	$i\sqrt{2}$	$-i\sqrt{2}$
0	0	0	0	0	0	0
$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0
$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0
$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0
ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI

;8	192 elements
T	
T	
'T	
ƦST	
'ST	
:ST	
Γ³	$P^4 = Q^4 = R^3 = S^2 = T^4 = E$
T³	$P^2 = Q^2$
T³	$QP = P^3Q$; $RP = QR$; $RQ = PQR$;
ST³	$SP = P^2QS$; $SQ = P^3S$; $SR = R^2S$;
ST³	$TP = PT$; $TQ = QT$; $TR = RT$;
ST³	$TS = ST^3$
	$\alpha = +1; \beta = +1$
	$\alpha = -1; \beta = +1$
	$\alpha = +1; \beta = -1$
	$\alpha = -1; \beta = -1$

›I	$A^2 = B^2 = C^3 = D^2 = I^2 = E$
	$BA = AB$; $CA = BC$; $CB = ABC$;
DI	$DA = BD$; $DB = AD$; $DC = C^2D$;
	$IA = AI$; $IB = BI$; $IC = CI$;
²DI	$ID = DI$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

	$1\epsilon_1$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$6\epsilon_4$	$8\epsilon_3$	$8\epsilon_3$	$24\epsilon_4$	$12\epsilon_8$
$_3(O_h)$	E	P^2	S^2	P^2S^2	P, P^3 Q, P^2Q PQ, P^3Q	PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^2R, P^2R^2	R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^2RS, P^2RS^3 P^2S, P^2S^3 P^3QS, P^3QS^3 $P^2R^2S, P^2R^2S^3$ $P^3R^2S, P^3R^2S^3$ P^2QRS, P^2QRS^3 RS, RS^3	PS P^2QS $PQRS$ P^3RS QR^2S P^3QR^2S PS^3 P^2QS^3 $PQRS^3$ P^3RS^3 QR^2S^3 $P^3QR^2S^3$
A_{1g}	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	1	1	1	-1	-1
E_g	2	2	2	2	2	-1	-1	0	0
T_{1g}	3	3	3	3	-1	0	0	-1	1
T_{2g}	3	3	3	3	-1	0	0	1	-1
A_{1u}	1	1	1	1	1	1	1	1	1
A_{2u}	1	1	1	1	1	1	1	-1	-1
E_u	2	2	2	2	2	-1	-1	0	0
T_{1u}	3	3	3	3	-1	0	0	-1	1
T_{2u}	3	3	3	3	-1	0	0	1	-1
$G'_{\alpha g}$	4	-4	4	-4	0	-1	1	0	0
$G'_{\alpha u}$	4	-4	4	-4	0	-1	1	0	0
$\{G''^{+}_{\alpha g}$	2	-2	2	-2	0	1	-1	0	$\mathrm{i}\sqrt{2}$
$\{G''^{-}_{\alpha g}$	2	-2	2	-2	0	1	-1	0	$-\mathrm{i}\sqrt{2}$
$\{G''^{+}_{\alpha u}$	2	-2	2	-2	0	1	-1	0	$\mathrm{i}\sqrt{2}$
$\{G''^{-}_{\alpha u}$	2	-2	2	-2	0	1	-1	0	$-\mathrm{i}\sqrt{2}$
$E_{1\beta}$	2	2	-2	-2	2	2	2	0	0
$E_{2\beta}$	2	2	-2	-2	2	-1	-1	0	0
$E_{3\beta}$	2	2	-2	-2	2	-1	-1	0	0
I_β	6	6	-6	-6	-2	0	0	0	0
$G_{1\alpha\beta}$	4	-4	-4	4	0	2	-2	0	0
$G_{2\alpha\beta}$	4	-4	-4	4	0	-1	1	0	0
$G_{3\alpha\beta}$	4	-4	-4	4	0	-1	1	0	0
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 3 (*cont.*)

$12\epsilon_8$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_6$	$2\epsilon_4$	$2\epsilon_4$	$12\epsilon_4$
QS^3 P^3S^3 PRS^3 P^3QRS^3 PQR^2S $P^2QR^2S^3$ QS P^3S PRS P^3QRS PQR^2S P^2QR^2S	PS^2, P^3S^2 QS^2, P^2QS^2 PQS^2, P^3QS^2	$PRS^2, P^3QR^2S^2$ $QRS^2, P^3R^2S^2$ $PQRS^2, P^2QR^2S^2$ $P^2RS^2, P^2R^2S^2$	R^2S^2, RS^2 PR^2S^2, P^2QRS^2 QR^2S^2, P^3QRS^2 PQR^2S^2, P^3RS^2	T, S^2T	P^2T, P^2S^2T	PT, P^3S^2T QT, P^2QS^2 PQT, P^3QS PS^2T, P^3T QS^2T, P^2Q PQS^2T, P^3Q
1	1	1	1	1	1	1
−1	1	1	1	1	1	1
0	2	−1	−1	2	2	2
1	−1	0	0	3	3	−1
−1	−1	0	0	3	3	−1
1	1	1	1	−1	−1	−1
−1	1	1	1	−1	−1	−1
0	2	−1	−1	−2	−2	−2
1	−1	0	0	−3	−3	1
−1	−1	0	0	−3	−3	1
0	0	−1	1	4	−4	0
0	0	−1	1	−4	4	0
$-i\sqrt{2}$	0	1	−1	2	−2	0
$i\sqrt{2}$	0	1	−1	2	−2	0
$-i\sqrt{2}$	0	1	−1	−2	2	0
$i\sqrt{2}$	0	1	−1	−2	2	0
0	−2	−2	−2	0	0	0
0	−2	1	1	0	0	0
0	−2	1	1	0	0	0
0	2	0	0	0	0	0
0	0	−2	2	0	0	0
0	0	1	−1	0	0	0
0	0	1	−1	0	0	0
BD *ACD* ABC^2D				*I*		*AI* *BI* *ABI*

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

	$8\epsilon_{12}$	$8\epsilon_{12}$	$8\epsilon_{12}$	$8\epsilon_{12}$	$24\epsilon_4$
$\ldots^2T$ $\ldots S^2T$ $\ldots S^2T$ $\ldots^3T$ $\ldots QT$ $\ldots^3QT$	$PRT, P^3QR^2S^2T$ $QRT, P^3R^2S^2T$ $PQRT, P^2QR^2S^2T$ $P^2RT, P^2R^2S^2T$	PRS^2T, P^3QR^2T QRS^2T, P^3R^2T $PQRS^2T, P^2QR^2T$ P^2RS^2T, P^2R^2T	R^2T, RS^2T PR^2T, P^2QRS^2T QR^2T, P^3QRS^2T PQR^2T, P^3RS^2T	R^2S^2T, RT PR^2S^2T, P^2QRT QR^2S^2T, P^3QRT PQR^2S^2T, P^3RT	ST, S^3T $PQST, PQS^3T$ R^2ST, R^2S^3T $PR^2ST, PR^2S^3\ldots$ $QRST, QRS^3T$ $P^2RST, P^2RS^3\ldots$ P^2ST, P^2S^3T P^3QST, P^3QS^3T $P^2R^2ST, P^2R^2S\ldots$ $P^3R^2ST, P^3R^2S\ldots$ $P^2QRST, P^2QR\ldots$ RST, RS^3T
	1	1	1	1	1
	1	1	1	1	−1
	−1	−1	−1	−1	0
	0	0	0	0	−1
	0	0	0	0	1
	−1	−1	−1	−1	−1
	−1	−1	−1	−1	1
	1	1	1	1	0
	0	0	0	0	1
	0	0	0	0	−1
	−1	−1	1	1	0
	1	1	−1	−1	0
	1	1	−1	−1	0
	1	1	−1	−1	0
	−1	−1	1	1	0
	−1	−1	1	1	0
	0	0	0	0	0
	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0
	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0
	0	0	0	0	0
	0	0	0	0	0
	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0
	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0
	ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$24\epsilon_4$	$12\epsilon_8$	$12\epsilon_8$	192 elements
T, S^3T	PST	QS^3T	
ST, PQS^3T	P^2QST	P^3S^3T	
ST, R^2S^3T	$PQRST$	P^3RS^3T	
ST, PR^2S^3T	P^3RST	P^3QRS^3T	
ST, QRS^3T	QR^2ST	PQR^2S^3T	
ST, P^2RS^3T	P^3QR^2ST	P^2QR^2ST	
ST, P^2S^3T	PS^3T	QST	$P^4 = Q^4 = R^3 = S^4 = T^4 = E$
T, P^3QS^3T	P^2QS^3T	P^3ST	$P^2 = Q^2;\ S^2 = T^2$
$ST, P^2R^2S^3T$	$PQRS^3T$	P^3RST	$QP = P^3Q;\ RP = QR;\ RQ = PQR$
$ST, P^3R^2S^3T$	P^3RS^3T	P^3QRST	$SP = P^2QS;\ SQ = P^3S;\ SR = R^2S$
ST, P^2QRS^3T	QR^2S^3T	PQR^2ST	$TP = PT;\ TQ = QT;\ TR = RT$
ST, RS^3T	P^3QR^2ST	$P^2QR^2S^3T$	$TS = S^3T$
1	1	1	
−1	−1	−1	
0	0	0	
−1	1	1	
1	−1	−1	$\alpha = +1;\ \beta = +1$
−1	−1	−1	
1	1	1	
0	0	0	
1	−1	−1	
−1	1	1	
0	0	0	
0	0	0	
0	$i\sqrt{2}$	$-i\sqrt{2}$	$\alpha = -1;\ \beta = +1$
0	$-i\sqrt{2}$	$i\sqrt{2}$	
0	$-i\sqrt{2}$	$i\sqrt{2}$	
0	$i\sqrt{2}$	$-i\sqrt{2}$	
0	0	0	
0	0	0	$\alpha = +1;\ \beta = -1$
0	0	0	
0	0	0	
0	0	0	
0	0	0	$\alpha = -1;\ \beta = -1$
0	0	0	
DI	ADI	BDI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$
$ABDI$			$BA = AB;\ CA = BC;\ CB = ABC$
C^2DI	$ABCDI$	$ACDI$	$DA = BD;\ DB = AD;\ DC = C^2D;$
AC^2DI			$IA = AI;\ IB = BI;\ IC = CI;$
$BCDI$	BC^2DI	ABC^2DI	$ID = DI$
CDI			

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$\mathscr{R}_4(O_h)$	$1\epsilon_1$	$1\epsilon_2$	$1\epsilon_2$	$1\epsilon_2$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_3$	$24\epsilon_4$	
	E	P^2	S^2	P^2S^2	P, P^3 Q, P^2Q PQ, P^3Q	PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^2R, P^2R^2	R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^2RS, P^2RS^3 P^2S, P^2S^3 P^3QS, P^3QS^3 $P^2R^2S, P^2R^2S^3$ $P^3R^2S, P^3R^2S^3$ P^2QRS, P^2QRS^3 RS, RS^3	
A_{1g}	1	1	1	1	1	1	1	1	
A_{2g}	1	1	1	1	1	1	1	−1	
E_g	2	2	2	2	2	−1	−1	0	
T_{1g}	3	3	3	3	−1	0	0	−1	
T_{2g}	3	3	3	3	−1	0	0	1	
A_{1u}	1	1	1	1	1	1	1	1	
A_{2u}	1	1	1	1	1	1	1	−1	
E_u	2	2	2	2	2	−1	−1	0	
T_{1u}	3	3	3	3	−1	0	0	−1	
T_{2u}	3	3	3	3	−1	0	0	1	
G_α	4	−4	4	−4	0	2	−2	0	
K_α K_α^+	4	−4	4	−4	0	−1	1	0	
K_α^-	4	−4	4	−4	0	−1	1	0	
E_β	2	2	−2	−2	2	2	2	0	
G_β G_β^+	2	2	−2	−2	2	−1	−1	0	
G_β^-	2	2	−2	−2	2	−1	−1	0	
I_β	6	6	−6	−6	−2	0	0	0	
$G_{1\alpha\beta}$ $G_{1\alpha\beta}^+$	2	−2	−2	2	0	1	−1	0	−i
$G_{1\alpha\beta}^-$	2	−2	−2	2	0	1	−1	0	i
$G_{2\alpha\beta}$ $G_{2\alpha\beta}^+$	2	−2	−2	2	0	1	−1	0	i
$G_{2\alpha\beta}^-$	2	−2	−2	2	0	1	−1	0	−i
$K_{\alpha\beta}$ $K_{\alpha\beta}^+$	4	−4	−4	4	0	−1	1	0	
$K_{\alpha\beta}^-$	4	−4	−4	4	0	−1	1	0	
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

TABLE 3 (*cont.*)

$12\epsilon_8$	$12\epsilon_8$	$6\epsilon_4$	$8\epsilon_6$	$8\epsilon_6$	$2\epsilon_4$	$2\epsilon_4$	1
PS^3 P^2QS^3 $PQRS^3$ P^3RS^3 QR^2S^3 $P^3QR^2S^3$ PS P^2QS $PQRS$ P^3RS QR^2S P^3QR^2S	QS P^3S PRS P^3QRS PQR^2S P^2QR^2S QS^3 P^3S^3 PRS^3 P^3QRS^3 PQR^2S^3 $P^2QR^2S^3$	PS^2, P^3S^2 QS^2, P^2QS^2 PQS^2, P^3QS^2	$PRS^2, P^3QR^2S^2$ $QRS^2, P^3R^2S^2$ $PQRS^2, P^2QR^2S^2$ $P^2RS^2, P^2R^2S^2$	R^2S^2, RS^2 PR^2S^2, P^2QRS^2 QR^2S^2, P^3QRS^2 PQR^2S^2, P^3RS^2	S^2T T	P^2S^2T P^2T	P Q PQ PS QS PQ P P^2 P^3 P^3 P^2Q P^3Q
1	1	1	1	1	1	1	
−1	−1	1	1	1	1	1	
0	0	2	−1	−1	2	2	
1	1	−1	0	0	3	3	−
−1	−1	−1	0	0	3	3	−
1	1	1	1	1	−1	−1	−
−1	−1	1	1	1	−1	−1	−
0	0	2	−1	−1	−2	−2	−
1	1	−1	0	0	−3	−3	
−1	−1	−1	0	0	−3	−3	
0	0	0	2	−2	0	0	
0	0	0	−1	1	0	0	
0	0	0	−1	1	0	0	
0	0	−2	−2	−2	0	0	
0	0	−2	1	1	0	0	
0	0	−2	1	1	0	0	
0	0	2	0	0	0	0	
$-i\sqrt{2}$	$i\sqrt{2}$	0	−1	1	2i	−2i	
$i\sqrt{2}$	$-i\sqrt{2}$	0	−1	1	−2i	2i	
$i\sqrt{2}$	$-i\sqrt{2}$	0	−1	1	2i	−2i	
$-i\sqrt{2}$	$i\sqrt{2}$	0	−1	1	−2i	2i	
0	0	0	1	−1	4i	−4i	
0	0	0	1	−1	−4i	4i	
AD	*BD*				*I*		*A*
							I
ABCD	*ACD*						*A*
BC^2D	ABC^2D						

$12\epsilon_2$	$8\epsilon_{12}$	$8\epsilon_{12}$	$8\epsilon_{12}$	$8\epsilon_{12}$	$24\epsilon_4$	$12\epsilon_8$
PT					ST, P^2ST	
QT					$PQST, P^3QST$	
PQT					R^2ST, P^2R^2ST	
PS^2T					PR^2ST, P^3R^2ST	
QS^2T	PRT	PQR^2T	P^2R^2T	RT	$QRST, P^2QRST$	
PQS^2T	QRT	PR^2T	P^3R^2T	P^2QRT	P^2RST, RST	
P^3T	$PQRT$	QR^2T	P^2QR^2T	P^3QRT	S^3T, P^2S^3T	PST, P^2QST
P^2QT	P^2RT	R^2T	P^3QR^2T	P^3RT	PQS^3T, P^3QS^3T	$PQRST, P^3RST$
P^3QT	$P^2R^2S^2T$	RS^2T	PRS^2T	PQR^2S^2T	$R^2S^3T, P^2R^2S^3T$	QR^2ST, P^3QR^2ST
P^3S^2T	$P^3R^2S^2T$	P^2QRS^2T	QRS^2T	PR^2S^2T	$PR^2S^3T, P^3R^2S^3T$	PS^3T, P^2QS^3T
P^2QS^2T	$P^2QR^2S^2T$	P^3QRS^2T	$PQRS^2T$	QR^2S^2T	QRS^3T, P^2QRS^3T	$PQRS^3T, P^3RS^3T$
P^3QS^2T	$P^3QR^2S^2T$	P^3RS^2T	P^2RS^2T	R^2S^2T	P^2RS^3T, RS^3T	$QR^2S^3T, P^3QR^2S^3T$
1	1	1	1	1	1	1
1	1	1	1	1	−1	−1
2	−1	−1	−1	−1	0	0
−1	0	0	0	0	−1	1
−1	0	0	0	0	1	−1
−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	1	1
−2	1	1	1	1	0	0
1	0	0	0	0	1	−1
1	0	0	0	0	−1	1
0	0	0	0	0	0	0
0	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0
0	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0
0	0	0	0	0	0	0
0	$-i\sqrt{3}$	$i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0
0	$i\sqrt{3}$	$-i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0
0	0	0	0	0	0	0
0	i	$-i$	i	$-i$	0	$\sqrt{2}$
0	$-i$	i	$-i$	i	0	$\sqrt{2}$
0	i	$-i$	i	$-i$	0	$-\sqrt{2}$
0	$-i$	i	$-i$	i	0	$-\sqrt{2}$
0	$-i$	i	$-i$	i	0	0
0	i	$-i$	i	$-i$	0	0
AI	ACI		C^2I	CI	DI	ADI
BI	BCI		AC^2I		$ABDI$	$ABCDI$
ABI	$ABCI$		ABC^2I		C^2DI	BC^2DI
					AC^2DI	
					$BCDI$	
					CDI	

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	$12\epsilon_8$	192 elements
	QST, P^3ST	$P^4 = Q^4 = R^3 = S^4 = T^4 = E$
T	$PRST, P^3QRST$	$P^2 = Q^2 = T^2$
T	PQR^2ST, P^2QR^2ST	$QP = P^3Q$; $RP = QR$; $RQ = PQR$
	QS^3T, P^3S^3T	$SP = P^2QS$; $SQ = P^3S$; $SR = R^2S$
^3T	PRS^3T, P^3QRS^3T	$TP = PT$; $TQ = QT$; $TR = RT$
^3T	$PQR^2S^3T, P^2QR^2S^3T$	$TS = S^3T$
	1	$\alpha = +1; \beta = +1$
	-1	
	0	
	1	
	-1	
	-1	
	1	
	0	
	-1	
	1	
	0	$\alpha = -1; \beta = +1$
	0	
	0	
	0	$\alpha = +1; \beta = -1$
	0	
	0	
	0	
	$-\sqrt{2}$	$\alpha = -1; \beta = -1$
	$-\sqrt{2}$	
	$\sqrt{2}$	
	$\sqrt{2}$	
	0	
	0	
	BDI	$A^2 = B^2 = C^2 = D^2 = I^2 = E$
	$ACDI$	$BA = AB$; $CA = BC$; $CB = ABC$
	ABC^2DI	$DA = BD$; $DB = AD$; $DC = C^2D$;
		$IA = AI$; $IB = BI$; $IC = CI$;
		$ID = DI$

TABLE 3 (*cont.*)

$\mathscr{R}(I)$	$1\epsilon_1$	$1\epsilon_2$	$30\epsilon_4$	$20\epsilon_6$	$20\epsilon_3$	$12\epsilon_5$
	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q PV, P^3V PV^2, P^3V^2 PV^3, P^3V^3 PV^4, P^3V^4 PRV^2, P^3RV^2 $PR^2V^2, P^3R^2V^2$ $PQRV^4, P^3QRV^4$ $PQR^2V^3, P^3QR^2V^3$ RV, P^2RV $R^2V^4, P^2R^2V^4$ QRV^3, P^2QRV^3 QR^2V, P^2QR^2V	PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 PQR^2V, PR^2V^4 QV, R^2V^3 PQV^3, P^3RV^4 QR^2V^2, P^2QV^4 RV^2, P^3QRV^3 P^2R, P^2R^2 P^2QRV, P^3QV^2	R, R^2 PR^2, P^2QR PQR^2, P^3R QR^2, P^3QR $QV^4, P^2QR^2V^2$ PQV^2, QRV $PQRV^3, P^2RV^2$ PRV^4, P^3QV^3 $P^2QV, P^2R^2V^3$ $P^3R^2V^4, P^3QR^2V$	V, V^4 QV^2, P^2RV^4 $RV^3, P^2QR^2V^4$ $PQRV, P^2QV^3$ R^2V^2, QR^2V^3 P^2R^2V, P^3QRV^2
A	1	1	1	1	1	1
T_1	3	3	-1	0	0	Φ
T_2	3	3	-1	0	0	Φ^{-1}
G	4	4	0	1	1	-1
H	5	5	1	-1	-1	0
$E_{\frac{1}{2}}$	2	-2	0	1	-1	$-\Phi$
$E_{\frac{7}{2}}$	2	-2	0	1	-1	Φ^{-1}
$G_{\frac{3}{2}}$	4	-4	0	-1	1	-1
$I_{\frac{5}{2}}$	6	-6	0	0	0	1
I	E		A B AB AF AF^2 AF^3 AF^4 ACF^2 AC^2F^2 $ABCF^4$ ABC^2F^3 CF C^2F^4 BCF^3 BC^2F	AC BC ABC ABC^2F, AC^2F^4 BF, C^2F^3 ABF^3 BC^2F^2 CF^2	C, C^2 AC^2 ABC^2 BC^2 BF^4 ABF^2, BCF $ABCF^3$ ACF^4	F, F^4 BF^2 CF^3 $ABCF$ C^2F^2, BC^2F^3

$(\Phi = \frac{1}{2}(1+\sqrt{5});\ \Phi^{-1} = \frac{1}{2}(-1+\sqrt{5}).)$

(cont.)

	$12\epsilon_5$	$12\epsilon_{10}$	$12\epsilon_{10}$	120 elements
	V^2, V^3	QV^3, P^3QRV	PRV, QRV^4	
	$PQV, P^3QR^2V^2$	$R^2V, PQRV^2$	PR^2V^3, PQV^4	$P^4 = Q^4 = R^3 = V^5 = E$
4	QRV^2, PQR^2V^4	RV^4, P^2QV^2	PQR^2V^2, P^3QV	$Q^2 = P^2$
3	PRV^3, PR^2V	QR^2V^4, P^2RV^3	P^2V^2, P^2V^3	$QP = P^3Q$; $RP = QR$
	P^3RV, P^2QRV^4	P^2V, P^2V^4	$P^2QRV^2, P^3QR^2V^4$	$RQ = PQR$; $VP = PV^4$
2	$P^3R^2V^3, P^3QV^4$	$P^2R^2V^2, P^2QR^2V^3$	P^3R^2V, P^3RV^3	$VQ = QR^2V^2$; $VR = P^2R^2V^4$
	1	1	1	$\alpha = +1$
	Φ^{-1}	Φ	Φ^{-1}	
	Φ	Φ^{-1}	Φ	
	-1	-1	-1	
	0	0	0	
	$-\Phi^{-1}$	Φ	Φ^{-1}	$\alpha = -1$
	Φ	$-\Phi^{-1}$	$-\Phi$	
	1	1	-1	
	-1	-1	1	
	F^2, F^3	BF^3	ACF, BCF^4	$A^2 = B^2 = C^3 = F^5 = E$
	ABF	$C^2F, ABCF^2$	AC^2F^3, ABV^4	$BA = AB$; $CA = BC$
	BCF^2, ABC^2F^4	CF^4	ABC^2F^2	$CB = ABC$; $FA = AF^4$
	ACF^3, AC^2F	BC^2F^4		$FB = BC^2F^2$; $FC = C^2F^4$
				$F^2C = BF$

5).)

$\mathscr{R}_1(I_h) = \mathscr{R}(I) \times \{E, T\}$

(I_h)	$1\epsilon_1$ E	$1\epsilon_2$ P^2	$30\epsilon_4$ P, P^3 Q, P^2Q PQ, P^3Q PV, P^3V PV^2, P^3V^2 PV^3, P^3V^3 PV^4, P^3V^4 PRV^2, P^3RV^2 $PR^2V^2, P^3R^2V^2$ $PQRV^4, P^3QRV^4$ $PQR^2V^3, P^3QR^2V^3$ RV, P^2RV $R^2V^4, P^2R^2V^4$ QRV^3, P^2QRV^3 QR^2V, P^2QR^2V	$20\epsilon_6$ PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 PQR^2V, PR^2V^4 QV, R^2V^3 PQV^3, P^3RV^4 QR^2V^2, P^2QV^4 RV^2, P^3QRV^3 P^2R, P^2R^2 P^2QRV, P^3QV^2	$20\epsilon_3$ R, R^2 PR^2, P^2QR PQR^2, P^3R QR^2, P^3QR $QV^4, P^2QR^2V^2$ PQV^2, QRV $PQRV^3, P^2RV^2$ PRV^4, P^3QV^3 $P^2QV, P^2R^2V^3$ $P^3R^2V^4, P^3QR^2V$	$12\epsilon_5$ V, V^4 QV^2, P^2RV^4 $RV^3, P^2QR^2V^4$ $PQRV, P^2QV^3$ R^2V^2, QR^2V^3 P^2R^2V, P^3QRV^2	PQ QR P P^3R P^3R
A_g	1	1	1	1	1	1	
T_{1g}	3	3	-1	0	0	Φ	
T_{2g}	3	3	-1	0	0	Φ^{-1}	
G_g	4	4	0	1	1	-1	
H_g	5	5	1	-1	-1	0	
A_u	1	1	1	1	1	1	
T_{1u}	3	3	-1	0	0	Φ	
T_{2u}	3	3	-1	0	0	Φ^{-1}	
G_u	4	4	0	1	1	-1	
H_u	5	5	1	-1	-1	0	
$G_{1\alpha}^{+}$	2	-2	0	1	-1	$-\Phi$	
$G_{1\alpha}^{-}$	2	-2	0	1	-1	$-\Phi$	
$G_{2\alpha}^{+}$	2	-2	0	1	-1	Φ^{-1}	
$G_{2\alpha}^{-}$	2	-2	0	1	-1	Φ^{-1}	
K_{α}^{+}	4	-4	0	-1	1	-1	
K_{α}^{-}	4	-4	0	-1	1	-1	
O_{α}^{+}	6	-6	0	0	0	1	
O_{α}^{-}	6	-6	0	0	0	1	
I_h	E		A B AB AF AF^2 AF^3 AF^4 ACF^2 AC^2F^2 $ABCF^4$ ABC^2F^3 CF C^2F^4 BCF^3 BC^2F	AC BG ABC ABC^2F, AC^2F^4 BF, C^2F^3 ABF^3 BC^2F^2 CF^2	C, C^2 AC^2 ABC^2 BC^2 BF^4 ABF^2, BCF $ABCF^3$ ACF^4	F, F^4 BF^2 CF^3 $ABCF$ C^2F^2, BC^2F^3	BC AC

$(\Phi = \tfrac{1}{2}(1+\sqrt{5})$; $\Phi^{-1} = \tfrac{1}{2}(-1+\sqrt{5})$.)

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TABLE 3 (*cont.*)

$12\epsilon_5$	$12\epsilon_{10}$	$12\epsilon_{10}$	$1\epsilon_4$	$1\epsilon_4$	$30\epsilon_2$	$20\epsilon_{12}$
V^2, V^3 $QV, P^3QR^2V^2$ RV^2, PQR^2V^4 PRV^3, PR^2V $^3RV, P^2QRV^4$ $^3R^2V^3, P^3QV^4$	QV^3, P^3QRV $R^2V, PQRV^2$ RV^4, P^2QV^2 QR^2V^4, P^2RV^3 P^2V, P^2V^4 $P^2R^2V^2, P^2QR^2V^3$	PRV, QRV^4 PR^2V^3, PQV^4 PQR^2V^2, P^3QV P^2V^2, P^2V^3 $P^2QRV^2, P^3QR^2V^4$ P^3R^2V, P^3RV^3	T	P^2T	PT P^3T QT P^2QT PQT P^3QT PVT P^3VT PV^2T P^3V^2T PV^3T P^3V^3T PV^4T P^3V^4T PRV^2T P^3RV^2T PR^2V^2T $P^3R^2V^2T$ $PQRV^4T$ P^3QRV^4T PQR^2V^3T $P^3QR^2V^3T$ RVT P^2RVT R^2V^4T $P^2R^2V^4T$ QRV^3T P^2QRV^3T QR^2VT P^2QR^2VT	PRT QRT $PQRT$ PQR^2VT QVT PQV^3T QR^2V^2T RV^2T P^2RT P^2QRVT P^3QR^2T P^3R^2T P^2QR^2T PR^2V^4T R^2V^3T P^3RV^4T P^2QV^4T P^3QRV^3T P^2R^2T P^3QV^2T
1	1	1	1	1	1	1
Φ^{-1}	Φ	Φ^{-1}	3	3	−1	0
Φ	Φ^{-1}	Φ	3	3	−1	0
−1	−1	−1	4	4	0	1
0	0	0	5	5	1	−1
1	1	1	−1	−1	−1	−1
Φ^{-1}	Φ	Φ^{-1}	−3	−3	1	0
Φ	Φ^{-1}	Φ	−3	−3	1	0
−1	−1	−1	−4	−4	0	−1
0	0	0	−5	−5	−1	1
$-\Phi^{-1}$	Φ	Φ^{-1}	2i	−2i	0	i
$-\Phi^{-1}$	Φ	Φ^{-1}	−2i	2i	0	−i
Φ	$-\Phi^{-1}$	$-\Phi$	2i	−2i	0	i
Φ	$-\Phi^{-1}$	$-\Phi$	−2i	2i	0	−i
−1	1	1	4i	−4i	0	−i
−1	1	1	−4i	4i	0	i
1	−1	−1	6i	−6i	0	0
1	−1	−1	−6i	6i	0	0
F^2, F^3 ABF CF^2, ABC^2F^4 ACF^3, AC^2F	BF^3 $C^2F, ABCF^2$ CF^4 BC^2F^4	ACF, BCF^4 AC^2F^3, ABF^4 ABC^2F^2	I		AI BI ABI AFI AF^2I AF^3I AF^4I ACF^2I AC^2F^2I $ABCF^4I$ ABC^2F^3I CFI C^2F^4I BCF^3I BC^2FI	ACI BCI $ABCI$ ABC^2FI BFI ABF^3I BC^2F^2I CF^2I AC^2F^4I C^2F^3I

$20\epsilon_{12}$	$12\epsilon_{20}$	$12\epsilon_{20}$	$12\epsilon_{20}$	$12\epsilon_{20}$	240 elements
PQR^2T					
PR^2T					
QR^2T					
$P^3R^2V^4T$					
$P^2R^2V^3T$					
PRV^4T					
QV^4T					
$PQRV^3T$					
R^2T	VT	P^2V^4T	V^2T	P^2V^3T	
PQV^2T	V^4T	P^2VT	V^3T	P^2V^2T	
P^3RT	QV^2T	RV^4T	$PQVT$	PQR^2V^2T	
P^2QRT	P^2RV^4T	P^2QV^2T	$P^3QR^2V^2T$	P^3QVT	
P^3QRT	RV^3T	QR^2V^4T	QRV^2T	$P^3QR^2V^4T$	
P^3QR^2VT	$P^2QR^2V^4T$	P^2RV^3T	PQR^2V^4T	P^2QRV^2T	$P^4 = Q^4 = R^3 = V^5 = T^4 = E$
P^2QVT	$PQRVT$	QV^3T	PRV^3T	P^3R^2VT	$P^2 = Q^2 = T^2$
P^3QV^3T	P^2QV^3T	P^3QRVT	PR^2VT	P^3RV^3T	$QP = P^3Q$; $RP = QR$
$P^2QR^2V^2T$	R^2V^2T	$P^2QR^2V^3T$	P^3RVT	QRV^4T	$RQ = PQR$; $VP = PV^4$
P^2RV^2T	QR^2V^3T	$P^2R^2V^2T$	P^2QRV^4T	$PRVT$	$VQ = QR^2V^2$; $VR = P^2R^2V^4$
RT	P^2R^2VT	$PQRV^2T$	$P^3R^2V^3T$	PQV^4T	$TP = PT$; $TQ = QT$
$QRVT$	P^3QRV^2T	R^2VT	P^3QV^4T	PR^2V^3T	$TR = RT$; $TV = VT$
1	1	1	1	1	
0	Φ	Φ^{-1}	Φ	Φ^{-1}	
0	Φ^{-1}	Φ	Φ^{-1}	Φ	
1	-1	-1	-1	-1	
-1	0	0	0	0	$\alpha = +1$
-1	-1	-1	-1	-1	
0	$-\Phi$	$-\Phi^{-1}$	$-\Phi$	$-\Phi^{-1}$	
0	$-\Phi^{-1}$	$-\Phi$	$-\Phi^{-1}$	$-\Phi$	
-1	1	1	1	1	
1	0	0	0	0	
$-\mathrm{i}$	$-\mathrm{i}\Phi$	$\mathrm{i}\Phi$	$-\mathrm{i}\Phi^{-1}$	$\mathrm{i}\Phi^{-1}$	
i	$\mathrm{i}\Phi$	$-\mathrm{i}\Phi$	$\mathrm{i}\Phi^{-1}$	$-\mathrm{i}\Phi^{-1}$	
$-\mathrm{i}$	$\mathrm{i}\Phi^{-1}$	$-\mathrm{i}\Phi^{-1}$	$\mathrm{i}\Phi$	$-\mathrm{i}\Phi$	
i	$-\mathrm{i}\Phi^{-1}$	$\mathrm{i}\Phi^{-1}$	$-\mathrm{i}\Phi$	$\mathrm{i}\Phi$	$\alpha = -1$
i	$-\mathrm{i}$	$-\mathrm{i}$	i	i	
$-\mathrm{i}$	i	i	$-\mathrm{i}$	$-\mathrm{i}$	
0	i	i	$-\mathrm{i}$	$-\mathrm{i}$	
0	$-\mathrm{i}$	$-\mathrm{i}$	i	i	
ABC^2I	FI	CF^4I	F^2I	ABC^2F^2I	$A^2 = B^2 = C^3 = F^5 = I^2 = E$
AC^2I	F^4I	BC^2F^4I	F^3I	BCF^4I	$BA = AB$; $CA = BC$
BC^2I	BF^2I	BF^3I	$ABFI$	$ACFI$	$CB = ABC$; $FA = AF^4$
ACF^4I	CF^3I	$ABCF^2I$	BCF^2I	ABF^4I	$FB = BC^2F^2$; $FC = C^2F^4$
BF^4I	$ABCFI$	C^2FI	ABC^2F^4I	AC^2F^3I	$F^2C = BF$
$ABCF^3$	C^2F^2I		ACF^3I		$IA = AI$; $IB = B$
C^2I	BC^2F^3I		AC^2FI		$IC = CI$; $IF = F$
ABF^2I					
CI					
$BCFI$					

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Table 3 (*cont.*)

$\mathscr{R}_1(K_h)$	E	R	∞C_∞^ϕ	S_2	σ_h	∞ elements
D_{jg}	$2j+1$	$2j+1$	$1+\sum_{l=1}^{l=j} 2\cos l\phi$	$2j+1$	$(-1)^j$	$\alpha=+1$
D_{ju}	$2j+1$	$2j+1$	$1+\sum_{l=1}^{l=j} 2\cos l\phi$	$-2j-1$	$(-1)^{j+1}$	
$D_{(j+\frac{1}{2})g}$	$2j+2$	$-2j-2$	$\sum_{l=0}^{l=j+1} 2\cos(l+\frac{1}{2})\phi$	$2j+2$	0	$\alpha=-1$
$D_{(j+\frac{1}{2})u}$	$2j+2$	$-2j-2$	$\sum_{l=0}^{l=j+1} 2\cos(l+\frac{1}{2})\phi$	$-2j-2$	0	
K_h	E		∞C_∞^ϕ	S_2	σ_h	

$\mathscr{R}_2(K_h)$	E	R	∞C_∞^ϕ	S_2	$\infty C_\infty^\phi S_2$	∞ elements
D_{jg}	$2j+1$	$2j+1$	$1+\sum_{l=1}^{l=j} 2\cos l\phi$	$2j+1$	$1+\sum_{l=0}^{l=j} 2\cos l\phi$	$\alpha=+1$
D_{ju}	$2j+1$	$2j+1$	$1+\sum_{l=1}^{l=j} 2\cos l\phi$	$-2j-1$	$1-\sum_{l=1}^{l=j} 2\cos l\phi$	
$D_{(j+\frac{1}{2})\alpha}$: $D^+_{(j+\frac{1}{2})\alpha}$	$2j+2$	$-2j-2$	$\sum_{l=0}^{l=j+1} 2\cos(l+\frac{1}{2})\phi$	$2\mathrm{i}(j+1)$	$\sum_{l=0}^{j+1} 2\mathrm{i}\cos(l+\frac{1}{2})\phi$	$\alpha=-1$
$D_{(j+\frac{1}{2})\alpha}$: $D^-_{(j+\frac{1}{2})\alpha}$	$2j+2$	$-2j-2$	$\sum_{l=0}^{l=j+1} 2\cos(l+\frac{1}{2})\phi$	$-2\mathrm{i}(j+1)$	$-\sum_{l=0}^{j+1} 2\mathrm{i}\cos(l+\frac{1}{2})\phi$	
K_h	E		∞C_∞^ϕ	S_2	$\infty C_\infty^\phi S_2$	

6. Ascent and descent in symmetry

If the point group G_2 is a subgroup of a point group G_1, the vector representations of G_2 can always be related to those of G_1 by a process of ascent and descent in symmetry (also known as induction and subduction) due to Frobenius (1898). Relations between the projective representations of G_1 and G_2 are in general, however, severely restricted, not only by differences in the multiplicators but also by the choice of representation group. Indeed for specific physical problems it may be advantageous to choose a particular representation group, and hence a particular set of projective representations, to facilitate the process of descent in symmetry.

To quote specific examples, the representation groups of O_h and D_{4h} are respectively of orders 192 and 128 and hence the projective representations of O_h cannot be subduced onto those of D_{4h} even though D_{4h} is a maximal subgroup of O_h. This is clearly because the multiplicator of D_{4h} is of greater order than that of O_h.

Further of the two representation groups of D_2, only $\mathscr{R}_1(D_2)$ is a subgroup of $\mathscr{R}(T)$ and hence there is clearly some advantage to be gained in dealing with the projective representations of D_2 derived from $\mathscr{R}_1(D_2)$ rather than those derived from $\mathscr{R}_2(D_2)$ when descent from the tetrahedral group is of interest.

Descents in symmetry are sometimes possible when the order of the multiplicator decreases from G_1 to G_2. For example, the multiplicator of O_h is of order 4 while those of O, T_d, T_h and D_{3d} are of order 2. However, only from $\mathscr{R}_1(O_h)$ and $\mathscr{R}_2(O_h)$ is a descent possible to a representation group of each of the four groups.

The only descents to maximal subgroups presented are those to maximal subgroups which are themselves representation groups of a point group. This includes cases where the multiplicator is necessarily trivial so that formally the point group is its own representation group. The correlations obey all of Frobenius's rules (1898): only descents have therefore been presented in the interests of economy of space. The consideration of different representation groups for a group G leads to more complete and detailed results than those obtainable by Harter (1969).

Table 4. Correlation of the irreducible representations of the C_{2nh} groups with those of their maximal subgroups

$\mathscr{R}_1(C_{4nh})$	C_{4n}
A_g	A
A_u	A
B_g	B
B_u	B
E_{lg}	E_l
E_{lu}	E_l
$E_{\frac{1}{2}n\alpha}$	$E_{\frac{1}{2}n\alpha}$
$E_{n\alpha}$	$A+B$
$G_{l\alpha}$	E_l+E_{2n-l}

$\mathscr{R}_1(C_{(4n-2)h})$	C_{4n-2}	$C_{(2n-1)h}$	$\mathscr{R}_1(C_{2h})$
A_g	A	A'	A_g
A_u	A	A'	A_u
B_g	B	A''	B_g
B_u	B	A''	B_u
E_{lg}	E_l	l odd: $E''_{n-\frac{1}{2}l-\frac{1}{2}}$ l even: $E'_{\frac{1}{2}l}$	$2B_g$ $2A_g$
E_{lu}	E_l	l odd: $E''_{n-\frac{1}{2}l-\frac{1}{2}}$ l even: $E'_{\frac{1}{2}l}$	$2B_u$ $2A_u$
$E_{n\alpha}$	$A+B$	$A'+A''$	$E_{1\alpha}$
$G_{l\alpha}$	E_l+E_{2n-l-1}	$E'_l+E''_l$	$2E_{1\alpha}$

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TABLE 4 (*cont.*)

$\mathscr{R}_2(C_{4nh})$	C_{4n}
A_g	A
A_u	A
B_g	A
B_u	A
$l \neq n$; E_{lg}	$E_{2n-\|2n-2l\|}$
$l \neq n$; E_{lu}	$E_{2n-\|2n-2l\|}$
E_{ng}	$2B$
E_{nu}	$2B$
$G_{l\alpha}$	$2E_{2l-1}$

$\mathscr{R}_2(C_{(4n-2)h})$	C_{4n-2}	$C_{(2n-1)h}$	$\mathscr{R}_2(C_{2n})$
A_g	A	A'	A_g
A_u	A	A'	A_u
B_g	A	A'	B_g
B_u	A	A'	B_u
E_{lg}	$E_{2n-1-\|2n-2l-1\|}$	$E'_{n-\frac{1}{2}-\|n-\frac{1}{2}-l\|}$	l odd: $2B_g$; l even: $2A_g$
E_{lu}	$E_{2n-1-\|2n-2l-1\|}$	$E'_{n-\frac{1}{2}-\|n-\frac{1}{2}-l\|}$	l odd: $2B_u$; l even: $2A_u$
E_α	$2B$	$2A''$	E_α
$G_{l\alpha}$	$2E_{2l-1}$	$2E''_l$	$2E_\alpha$

TABLE 5. CORRELATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE REPRESENTATION GROUPS OF THE DIHEDRAL GROUPS D_{2n} WITH THOSE OF THEIR MAXIMAL SUBGROUPS

$\mathscr{R}_1(D_{2n})$	C_{2n}	n odd ($\neq 1$) $\mathscr{R}_1(D_2)$	n even $\mathscr{R}_1(D_n)$
A_1	A	A_1	A_1
A_2	A	A_2	A_2
B_1	A	B_1	A_1
B_2	A	B_2	A_2
$l \neq \frac{1}{2}n$; E_l	$E_{n-\|n-2l\|}$	l odd: B_1+B_2; l even: A_1+A_2	$E_{\frac{1}{2}n-\|\frac{1}{2}n-l\|}$
$E_{\frac{1}{2}n}$ or $E_{(\frac{1}{2}n+\frac{1}{2})\alpha}$	$2B$	$E_{1\alpha}$	B_1+B_2
$l \neq \frac{1}{2}n+\frac{1}{2}$; $E_{l\alpha}$	$E_{n-\|n-(2l-1)\|}$	$E_{1\alpha}$	$E_{\{n+\frac{1}{2}-\|n+\frac{1}{2}-l\|\}\alpha}$

$\mathscr{R}_2(D_{2n})$	C_{2n}	n odd D_n	n even $\mathscr{R}_2(D_n)$	n odd ($\neq 1$) $\mathscr{R}_2(D_2)$
A_1	A	A_1	A_1	A_1
A_2	A	A_2	A_2	A_2
B_1	A	A_1	A_1	B_1
B_2	A	A_2	A_2	B_2
$l \neq \frac{1}{2}n$; E_l	$E_{n-\|n-2l\|}$	$E_{\frac{1}{2}n-\|\frac{1}{2}n-2l \bmod n\|}$	$E_{\frac{1}{2}n-\|\frac{1}{2}n-l\|}$	l odd: B_1+B_2; l even: A_1+A_2
n even; $E_{\frac{1}{2}n}$	$2B$		B_1+B_2	
$l \neq \frac{1}{2}n+\frac{1}{2}$; $E_{l\alpha}$	$E_{n-\|n-(2l-1)\|}$	$E_{\frac{1}{2}n-\|\frac{1}{2}n-(2l-1) \bmod n\|}$	$E_{\{n+\frac{1}{2}-\|n+\frac{1}{2}-l\|\}\alpha}$	$E_{1\alpha}$
n odd; $E_{(\frac{1}{2}n+\frac{1}{2})\alpha}$	$E_{n-\|n-(2l-1)\|}$	A_1+A_2		$E_{1\alpha}$

$\mathscr{R}_3(D_{4n-2})$	C_{4n-2}	D_{2n-1}	$n \neq 1$ $\mathscr{R}_2(D_2)$
A_1	A	A_1	A_1
A_2	A	A_2	B_1
B_1	B	A_2	A_2
B_2	B	A_1	B_2
E_l	E_l	$E_{n-\frac{1}{2}-\|n-l-\frac{1}{2}\|}$	l odd: A_2+B_2; l even: A_1+B_1
E_α	$A+B$	A_1+A_2	$E_{1\alpha}$
$G_{l\alpha}$	$E_{2l}+E_{2n-2l-1}$	$2E_{n-\frac{1}{2}-\|n-\frac{1}{2}-2l\|}$	$2E_{1\alpha}$

$\mathscr{R}_3(D_{4n})$	C_{4n}	$\mathscr{R}_1(D_{2n})$	$\mathscr{R}_2(D_{2n})$
A_1	A	A_1	A_1
A_2	A	A_2	A_2
B_1	A	A_2	A_1
B_2	A	A_1	A_2
$l \neq n$; E_l	$E_{2n-\|2n-2l\|}$	$E_{n-\|n-l\|}$	$E_{n-\|n-l\|}$
E_n	$2B$	B_1+B_2	B_1+B_2
$G_{l\alpha}$	$2E_{2n-\|2n-2l+1\|}$	$2E_{l\alpha}$	$2E_{l\alpha}$

Table 6. Correlation of the irreducible representations of the representation groups of the D_{2nh} groups with those of their maximal subgroups

(The D_{4nh} groups have no representation groups as maximal subgroups.)

$\mathscr{R}_1(D_{(4n+2)h})$	$\mathscr{R}(D_{2n})$
A_{1g}	A_{1g}
A_{1u}	A_{1u}
A_{2g}	A_{2g}
A_{2u}	A_{2u}
B_{1g}	B_{1g}
B_{1u}	B_{1u}
B_{2g}	B_{2g}
B_{2u}	B_{2u}
E_{lg}	l odd: $B_{1g}+B_{2g}$ l even: $A_{1g}+A_{2g}$
E_{lu}	l odd: $B_{1u}+B_{2u}$ l even: $A_{1u}+A_{2u}$
$E_{l\alpha}$	l odd: $E_{1\alpha}$ l even: $E_{2\alpha}$
$E_{1\beta}$	$E_{1\beta}$
$E_{2\beta}$	$E_{2\beta}$
$G_{l\beta}$	l odd: $2E_{2\beta}$ l even: $2E_{1\beta}$
$E_{1\gamma}$	$E_{1\gamma}$
$E_{2\gamma}$	$E_{2\gamma}$
$G_{l\gamma}$	$E_{1\gamma}+E_{2\gamma}$
$G_{l\alpha\beta}$	$G_{1\alpha\beta}$
$G_{\alpha\gamma}$	$G_{\alpha\gamma}$
$G_{l\alpha\gamma}$	$G_{\alpha\gamma}$
$G_{\gamma\beta}$	$G_{\gamma\beta}$
$G_{l\gamma\beta}$	$G_{\gamma\beta}$
$E_{1\alpha\beta\gamma}$	$E_{1\alpha\beta\gamma}$
$E_{2\alpha\beta\gamma}$	$E_{2\alpha\beta\gamma}$
$G_{l\alpha\beta\gamma}$	$E_{1\alpha\beta\gamma}+E_{2\alpha\beta\gamma}$

$\mathscr{R}_2(D_{(4n+2)h})$	$\mathscr{R}(D_{2h})$
A_{1g}	A_{1g}
A_{1u}	A_{1u}
A_{2g}	A_{2g}
A_{2u}	A_{2u}
B_{1g}	B_{1g}
B_{1u}	B_{1u}
B_{2g}	B_{2g}
B_{2u}	B_{2u}
E_{lg}	l odd: $B_{1u}+B_{2u}$ l even: $A_{1g}+A_{2g}$
E_{lu}	l odd: $B_{1g}+B_{2g}$ l even: $A_{1u}+A_{2u}$
$E_{1\alpha}$	$E_{1\alpha}$
$E_{2\alpha}$	$E_{2\alpha}$
$G_{l\alpha}$	$E_{1\alpha}+E_{2\alpha}$
$E_{1\beta}$	$E_{1\beta}$
$E_{2\beta}$	$E_{2\beta}$
$G_{l\beta}$	$E_{1\beta}+E_{2\beta}$
$E_{1\gamma}$	$E_{1\gamma}$
$E_{2\gamma}$	$E_{2\gamma}$
$G_{l\gamma}$	$E_{1\gamma}+E_{2\gamma}$
$G_{l\alpha\beta}$	$G_{1\alpha\beta}$
$E_{\alpha\gamma}$	$E_{\alpha\gamma}$
$G_{l\alpha\gamma}$	$2E_{\alpha\gamma}$
$E_{\gamma\beta}$	$E_{\gamma\beta}$
$G_{l\gamma\beta}$	$2E_{\gamma\beta}$
$E_{l\alpha\beta\gamma}$	l odd: $E_{1\alpha\beta\gamma}$ l even: $E_{2\alpha\beta\gamma}$

Table 7. Correlation of the irreducible representations of the representation groups of the tetrahedral groups with their maximal subgroups

(The two representation groups of the regular tetrahedral group (T_d) are isomorphic with those of the octahedral rotation group (O), q.v. The tables for $\mathscr{R}_1(O)$ and $\mathscr{R}_2(O)$ should therefore be used, with the corresponding changes in the subgroups, viz. $\mathscr{R}(D_4) \rightarrow \mathscr{R}(D_{2d})$ and $D_3 \rightarrow C_{3v}$.)

$\mathscr{R}(T)$	$\mathscr{R}_1(D_2)$	C_3
A	A_1	A
E	$2A_1$	E
T	$A_2+B_1+B_2$	$A+E$
$E_{\frac{1}{2}}$	$E_{1\alpha}$	E
$G_{\frac{3}{2}}$	$2E_{1\alpha}$	$2A+E$

$\mathscr{R}_1(T_h)$	$\mathscr{R}(T)$	S_6
A_g	A	A_g
A_u	A	A_u
E_g	E	E_g
E_u	E	E_u
T_g	T	A_g+E_g
T_u	T	A_u+E_u
$E_{\frac{1}{2}g}$	$E_{\frac{1}{2}}$	E_g
$E_{\frac{1}{2}u}$	$E_{\frac{1}{2}}$	E_u
$G_{\frac{3}{2}g}$	$G_{\frac{3}{2}}$	$2A_g+E_g$
$G_{\frac{3}{2}u}$	$G_{\frac{3}{2}}$	$2A_u+E_u$

$\mathscr{R}_2(T_h)$	$\mathscr{R}(T)$	S_6
A_g	A	A_g
A_u	A	A_g
E_g	E	E_g
E_u	E	E_g
T_g	T	A_g+E_g
T_u	T	A_g+E_g
G_α	$2E_{\frac{1}{2}}$	$2E_u$
G'_α	$G_{\frac{3}{2}}$	$2A_u+E_u$
G''_α	$G_{\frac{3}{2}}$	$2A_u+E_u$

TABLE 8. CORRELATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE REPRESENTATION GROUPS OF THE OCTAHEDRAL GROUPS WITH THOSE OF THEIR MAXIMAL SUBGROUPS

(Entries for the subgroups $\mathscr{R}_1(T_d)$ and $\mathscr{R}_2(T_d)$ are identical with those for the isomorphic groups $\mathscr{R}_1(O)$ and $\mathscr{R}_2(O)$.)

$\mathscr{R}_1(O)$	$\mathscr{R}(T)$	$\mathscr{R}_1(D_4)$
A_1	A	A_1
A_2	A	B_2
E	E	A_1+B_2
T_1	T	A_2+E
T_2	T	B_1+E
$E_{\frac{1}{2}}$	$E_{\frac{1}{2}}$	$E_{1\alpha}$
$E_{\frac{5}{2}}$	$E_{\frac{1}{2}}$	$E_{2\alpha}$
$G_{\frac{3}{2}}$	$G_{\frac{3}{2}}$	$E_{1\alpha}+E_{2\alpha}$

$\mathscr{R}_2(O)$	$\mathscr{R}(T)$	$\mathscr{R}_3(D_4)$	D_3
A_1	A	A_1	A_1
A_2	A	B_2	A_2
E	E	A_1+B_2	E
T_1	T	A_2+E_1	A_2+E
T_2	T	B_1+E_1	A_1+E
$G_{\frac{1}{2}}$	$2E_{\frac{1}{2}}$	$E_{1\alpha}$	$2E$
$G_{\frac{3}{2}}$	$G_{\frac{3}{2}}$	$E_{2\alpha}$	A_1+A_2+E

$\mathscr{R}_1(O_h)$	$\mathscr{R}_1(O)$	$\mathscr{R}_1(T_h)$	$\mathscr{R}_3(D_{3d})$
A_{1g}	A_1	A_g	A_1
A_{1u}	A_2	A_u	B_1
A_{2g}	A_2	A_g	A_2
A_{2u}	A_2	A_u	B_2
E_g	E	E_g	E_2
E_u	E	E_u	E_1
T_{1g}	T_1	T_g	A_2+E_2
T_{1u}	T_1	T_u	B_2+E_1
T_{2g}	T_2	T_g	A_1+E_2
T_{2u}	T_2	T_u	B_1+E_1
G_α	$E_{\frac{1}{2}}+E_{\frac{5}{2}}$	$2E_{\frac{1}{2}g}$	$G_{1\alpha}$
K_α	$2G_{\frac{3}{2}}$	$2G_{\frac{3}{2}g}$	$2E_\alpha+G_{1\alpha}$
E_β	A_1+A_2	$2A_u$	E_α
G_β	$2E$	$2E_u$	$G_{1\alpha}$
I_β	T_1+T_2	$2T_u$	$E_\alpha+G_{1\alpha}$
$E'_{\alpha\beta g}$	$E_{\frac{1}{2}}$	$E_{\frac{1}{2}u}$	E_2
$E'_{\alpha\beta u}$	$E_{\frac{5}{2}}$	$E_{\frac{1}{2}u}$	E_1
$E''_{\alpha\beta g}$	$E_{\frac{5}{2}}$	$E_{\frac{1}{2}u}$	E_2
$E''_{\alpha\beta u}$	$E_{\frac{1}{2}}$	$E_{\frac{1}{2}u}$	E_1
$G_{\alpha\beta g}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}u}$	$A_1+A_2+E_2$
$G_{\alpha\beta u}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}u}$	$B_1+B_2+E_1$

$\mathscr{R}_2(O_h)$	$\mathscr{R}_2(O)$	$\mathscr{R}_1(T_h)$	$\mathscr{R}_2(D_{3d})$
A_{1g}	A_1	A_g	A_1
A_{1u}	A_1	A_g	B_1
A_{2g}	A_2	A_g	A_2
A_{2u}	A_2	A_g	A_2
E_g	E	E_g	E_2
E_u	E	E_g	E_1
T_{1g}	T_1	T_g	A_2+E_2
T_{1u}	T_1	T_g	B_2+E_1
T_{2g}	T_2	T_g	A_1+E_2
T_{2u}	T_2	T_g	B_1+E_1
$G'_{\alpha g}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}g}$	$A_1+A_2+E_2$
$G'_{\alpha u}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}g}$	$B_1+B_2+E_1$
$G''_{\alpha g}$	$G_{\frac{1}{2}}$	$2E_{\frac{1}{2}g}$	$2E_2$
$G''_{\alpha u}$	$G_{\frac{1}{2}}$	$2E_{\frac{1}{2}g}$	$2E_1$
$E_{1\beta}$	A_1+A_2	$2A_u$	$E_{2\alpha}$
$E_{2\beta}$	E	E_u	$E_{3\alpha}$
$E_{3\beta}$	E	E_u	$E_{1\alpha}$
I_β	T_1+T_2	$2T_u$	$E_{1\alpha}+E_{2\alpha}+E_{3\alpha}$
$G_{1\alpha\beta}$	$G_{\frac{1}{2}}$	$2E_{\frac{1}{2}u}$	$E_{1\alpha}+E_{3\alpha}$
$G_{2\alpha\beta}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}u}$	$E_{1\alpha}+E_{2\alpha}$
$G_{3\alpha\beta}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}u}$	$E_{2\alpha}+E_{3\alpha}$

$\mathscr{R}_3(O_h)$	$\mathscr{R}_1(T_h)$	$\mathscr{R}_1(D_{3d})$
A_{1g}	A_g	A_1
A_{1u}	A_g	B_1
A_{2g}	A_g	A_2
A_{2u}	A_g	B_2
E_g	E_g	E_2
E_u	E_g	E_1
T_{1g}	T_g	A_2+E_2
T_{1u}	T_g	B_1+E_1
T_{2g}	T_g	A_1+E_2
T_{2u}	T_g	B_2+E_1
$G'_{\alpha g}$	$G_{\frac{3}{2}g}$	$B_1+B_2+E_1$
$G'_{\alpha u}$	$G_{\frac{3}{2}g}$	$A_1+A_2+E_2$
$G''_{\alpha g}$	$2E_{\frac{1}{2}g}$	$2E_1$
$G''_{\alpha u}$	$2E_{\frac{1}{2}g}$	$2E_2$
$E_{1\beta}$	$2A_u$	$E_{2\alpha}$
$E_{2\beta}$	E_u	$E_{3\alpha}$
$E_{3\beta}$	E_u	$E_{1\alpha}$
I_β	$2T_u$	$E_{1\alpha}+E_{2\alpha}+E_{3\alpha}$
$G_{1\alpha\beta}$	$E_{\frac{1}{2}u}$	$E_{1\alpha}+E_{3\alpha}$
$G_{2\alpha\beta}$	$G_{\frac{3}{2}u}$	$E_{2\alpha}+E_{3\alpha}$
$G_{3\alpha\beta}$	$G_{\frac{3}{2}u}$	$E_{1\alpha}+E_{2\alpha}$

$\mathscr{R}_4(O_h)$	$\mathscr{R}_1(T_h)$	$\mathscr{R}_2(T_h)$
A_{1g}	A_g	A_g
A_{1u}	A_g	A_u
A_{2g}	E_g	A_g
A_{2u}	T_g	A_u
E_g	T_g	E_g
E_u	A_g	E_u
T_{1g}	A_g	T_g
T_{1u}	E_g	T_u
T_{2g}	T_g	T_g
T_{2u}	T_g	T_u
G_α	$2E_{\frac{1}{2}g}$	G_α
K_α	$2G_{\frac{3}{2}g}$	$G'_\alpha+G''_\alpha$
E_β	$2A_u$	A_g+A_u
G_β	$2E_u$	E_g+E_u
I_β	$2T_u$	T_g+T_u
$G_{1\alpha\beta}$	$2E_{\frac{1}{2}u}$	G_α
$G_{2\alpha\beta}$	$G_{\frac{3}{2}u}$	G_α
$G_{3\alpha\beta}$	$2G_{\frac{3}{2}u}$	$G'_\alpha+G''_\alpha$

Table 9. Correlation of the irreducible representations of the representation groups of the icosahedral groups with those of their maximal subgroups

$\mathscr{R}(I)$	$\mathscr{R}(T)$	D_5	D_3
A	A	A_1	A_1
T_1	T	A_2+E_1	A_2+E
T_2	T	A_2+E_2	A_2+E
G	$A+T$	E_1+E_2	A_1+A_2+E
H	$E+T$	$A_1+E_1+E_2$	A_1+2E
$E_{\frac{1}{2}}$	$E_{\frac{1}{2}}$	E_2	E
$E_{\frac{7}{2}}$	$E_{\frac{1}{2}}$	E_1	E
$G_{\frac{3}{2}}$	$G_{\frac{3}{2}}$	E_1+E_2	A_1+A_2+E
$I_{\frac{5}{2}}$	$E_{\frac{1}{2}}+G_{\frac{3}{2}}$	$A_1+A_2+E_1+E_2$	A_1+A_2+2E

$\mathscr{R}_1(I_h)$	$\mathscr{R}(I)$	$\mathscr{R}_1(T_h)$
A_g	A	A_g
A_u	A	A_u
T_{1g}	T_1	T_g
T_{1u}	T_1	T_u
T_{2g}	T_2	T_g
T_{2u}	T_2	T_h
G_g	G	A_g+T_g
G_u	G	A_u+T_u
H_g	H	E_g+T_g
H_u	H	E_u+T_u
$E_{\frac{1}{2}g}$	$E_{\frac{1}{2}}$	$E_{\frac{1}{2}g}$
$E_{\frac{1}{2}u}$	$E_{\frac{1}{2}}$	$E_{\frac{1}{2}u}$
$E_{\frac{7}{2}g}$	$E_{\frac{7}{2}}$	$E_{\frac{1}{2}g}$
$E_{\frac{7}{2}u}$	$E_{\frac{7}{2}}$	$E_{\frac{1}{2}u}$
$G_{\frac{3}{2}g}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}g}$
$G_{\frac{3}{2}u}$	$G_{\frac{3}{2}}$	$G_{\frac{3}{2}u}$
$I_{\frac{5}{2}g}$	$I_{\frac{5}{2}}$	$E_{\frac{1}{2}g}+G_{\frac{3}{2}g}$
$I_{\frac{5}{2}u}$	$I_{\frac{5}{2}}$	$E_{\frac{1}{2}u}+G_{\frac{3}{2}u}$

$\mathscr{R}_2(I_h)$	$\mathscr{R}(I)$	$\mathscr{R}_2(T_h)$
A_g	A	A_g
A_u	A	A_u
T_{1g}	T_1	T_g
T_{1u}	T_1	T_u
T_{2g}	T_2	T_g
T_{2u}	T_2	T_u
G_g	G	A_g+T_g
G_u	G	A_u+T_u
H_g	H	E_g+T_g
H_u	H	E_u+T_u
$G_{1\alpha}$	$2E_{\frac{1}{2}}$	G_α
$G_{2\alpha}$	$2E_{\frac{7}{2}}$	G_α
K_α	$2G_{\frac{3}{2}}$	$G'_\alpha+G''_\alpha$
O_α	$2I_{\frac{5}{2}}$	$G_\alpha+G'_\alpha+G''_\alpha$

Table 10. Correlation of the irreducible representations of the representation groups of the spherical rotation-reflection group K_h with those of its maximal subgroups

$\mathscr{R}_1(K_h)$	$\mathscr{R}_1(I_h)$
D_{0g}	A_g
D_{0u}	A_u
D_{1g}	T_{1g}
D_{1u}	T_{1u}
D_{2g}	H_g
D_{2u}	H_u
D_{3g}	$T_{2g}+G_g$
D_{3u}	$T_{2u}+G_u$
....	....
$D_{\frac{1}{2}g}$	$E_{\frac{1}{2}g}$
$D_{\frac{1}{2}u}$	$E_{\frac{1}{2}u}$
$D_{\frac{3}{2}g}$	$G_{\frac{3}{2}g}$
$D_{\frac{3}{2}u}$	$G_{\frac{3}{2}u}$
$D_{\frac{5}{2}g}$	$I_{\frac{5}{2}g}$
$D_{\frac{5}{2}u}$	$I_{\frac{5}{2}u}$
$D_{\frac{7}{2}g}$	$E_{\frac{7}{2}g}+I_{\frac{5}{2}g}$
$D_{\frac{7}{2}u}$	$E_{\frac{7}{2}u}+I_{\frac{5}{2}u}$
....	....

$\mathscr{R}_2(K_h)$	$\mathscr{R}_2(I_h)$
D_{0g}	A_g
D_{0u}	A_u
D_{1g}	T_{1g}
D_{1u}	T_{1u}
D_{2g}	H_g
D_{2u}	H_u
D_{3g}	$T_{2g}+G_g$
D_{3u}	$T_{2u}+G_u$
....	....
$D_{\frac{1}{2}\alpha}$	$G_{1\alpha}$
$D_{\frac{3}{2}\alpha}$	K_α
$D_{\frac{5}{2}\alpha}$	O_α
$D_{\frac{7}{2}\alpha}$	$G_{2\alpha}+O_\alpha$
....	....

7. The symmetrized powers of projective representations

The direct product of projective representations has been considered by Rudra (1964) and corrected by Harter (1969). However, the resulting formulae are unwieldy because by not involving the actual representation groups they require a knowledge of the large numbers of factor systems of the projective representations and the formation of lengthy products of these.

The use of the standard formulae for vector representations in the representation group, however, enables the calculation to be performed for projective representations without reference to factor systems. Further, there are no complications or need for special theories in the calculation of the symmetrized powers of projective representations, which do not appear to have been considered hitherto. The cases of particular physical interest are those of the symmetrized squares and cubes which are used in calculating the expectation values of real and imaginary operators as well as in applying the Landau–Lifschitz theory of phase transitions. The results may be found on pages 134–148 of a thesis by one of us (Green 1976). The symmetrized powers of the vector representations of the representation groups are the same as those for the point groups and hence may be found in the papers of Jahn & Teller (1937) and Boyle (1972).

The fact that the powers of any representation of a group must be symmetrizable provides convincing proof of errors in the underived tables of projective representations published by Janssen (1973). By deducing the representation group from the projective representations published one can, by comparison with our tables, deduce the characters for those elements of the representation group which do not map onto G and hence perform a rigorous symmetrization – usually the symmetrization of the square is sufficient to reveal any discrepancy. In this way the characters of magnitude 2i in the projective representations Γ_{13} and Γ_{21} of D_{2h} were found to be actually 2 while the 2 in Γ_{15} should be 2i. Döring's (1956) and Hurley's (1966) projective representations for D_{2h} were similarly wrong since their projective representations only contain real characters.

The symmetrized powers of projective representations differ considerably according to the representation group chosen. However, in physical problems such as those to be discussed in the next two sections, there will always be one choice for which the set of projective characters is physically relevant *without modification* even though there may be phase factors in the gauge transformation. Hence by identifying this choice the above tables can be used to solve any given physical problem requiring symmetrized squares or cubes.

8. Applications

8.1 *Derivation of the double-valued representations of the point groups*

Projective representations may be used to find the double-valued representations of a group, irrespective of whether the multiplicator is of order 2 or not. It should be emphasized that whereas the representation group is the extension of M by G, the double group, G', is the extension of C_1' by G where C_1' is the group consisting of the identity and the element, R, which reverses the sign of the spin functions for systems with an odd number of electrons. The isomorphism of an $\mathscr{R}(G)$ with G' is therefore inherent when M is of order 2 and G is a non-Abelian point group. A certain class of representations of $\mathscr{R}(G)$, which corresponds to a class of projective representations of G, can always be modified so that they provide the double-valued representations of G and, further, these unique double-valued representations can be obtained from any of the different sets of projective representations corresponding to representation groups. The relation of double-valued representations to projective representations was first discussed by Weyl (1931) and subsequently developed by Hurley (1966).

The double-valued representations of a group G' are defined such that

$$\delta(Rg_i) = -\delta(g_i),$$

where R commutes with all elements g_i of G'. This law is also obeyed for that class α of representations of $\mathscr{R}(G)$ for which the representative matrices

$$\Delta(m_\alpha r_i) = -\Delta(r_i)$$

where m_α is an element of the multiplicator, since by projection into G, both $\pi(m_\alpha r_i) = g_i$ and $\pi(r_i) = g_i$ and, in general, $\Delta(r_i) = \phi\delta(g_i)$, where ϕ is a phase factor to be determined. The double-valued representations are thus identified by the class α of representations of $\mathscr{R}(G)$ and their character systems can be determined once the phase factor (known as a gauge transformation in this context) has been found by comparing the relationships between the generating matrices $\{P, Q\}$ which hold for the group $\mathscr{R}(G)$ with those between the generating matrices $\{A, B\}$ which hold for the double-valued representations of the group G'. This will now be illustrated in the case of the dihedral group $G = D_4$:

D_4'	$\mathscr{R}_1(D_4)$	$\mathscr{R}_2(D_4)$	$\mathscr{R}_3(D_4)$
$A^4 = -E$ $B^2 = -E$ $BA = -A^3B$	$P^4 = \alpha E$ $Q^2 = \alpha E$ $QP = \alpha P^3Q$	$P^4 = \alpha E$ $Q^2 = E$ $QP = \alpha P^3Q$	$P^4 = \alpha E$ $Q^2 = E$ $QP = P^3Q$
required gauge transformations	$\begin{cases} P \to A \\ Q \to B \end{cases}$	$P \to A$ $Q \to \pm iB$	$P \to \pm iA$, $Q \to \pm iB$
required class of representations	$\alpha = -1$	$\alpha = -1$	$\alpha = -1$

The character systems are now derived by effecting the gauge transformations on the elements of a representation group and then dividing the relevant projective characters through by any resulting phase factors to obtain the characters of the double-valued representations of D_4'. As an example we choose $\mathscr{R}_3(D_4)$. The required projective characters are those of the separably-degenerate $G_{1\alpha}$ representation:

$\mathscr{R}_3(D_4)$	E	P^4	$\{P\}$	$\{P^5\}$	$\{P^2\}$	$\{Q\}$	$\{PQ\}$
$G_{1\alpha}$ $G_{1\alpha}^+$	2	-2	$i\sqrt{2}$	$-i\sqrt{2}$	0	0	0
$G_{1\alpha}^-$	2	-2	$-i\sqrt{2}$	$i\sqrt{2}$	0	0	0
phase factor × D_4'	E	A^4	$i\{A\}$	$i\{A^5\}$	$-\{A^2\}$	$i\{B\}$	$-\{AB\}$
$E_{\frac{1}{2}}$	2	-2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0
$E_{\frac{3}{2}}$	2	-2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0

This process has, therefore, resolved the complex–conjugate pair of representations $\{G_{1\alpha}^+, G_{1\alpha}^-\}$ into the real double-valued representations $\{E_{\frac{1}{2}}, E_{\frac{3}{2}}\}$ of D_4'. The same representations are obtained as a set, whatever combinations of ± signs in the phase factors are used. Further, the same set of representations is similarly obtained from $\mathscr{R}_1(D_4)$ and $\mathscr{R}_2(D_4)$.

The case of the regular octahedral double group, $O_h' = G$, is interesting since it provides the simplest example among the point groups where the double-valued representations are derived from one of several classes of projective representations. The generating relationships for the matrices corresponding to the elements of the different representation groups are simplified by writing them in terms of the matrices of those elements which can be mapped onto matrices of corresponding elements of O_h':

O_h'	$\mathscr{R}_1(O_h)$	$\mathscr{R}_2(O_h)$	$\mathscr{R}_3(O_h)$	$\mathscr{R}_4(O_h)$
	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = T^2 = \alpha E$
$A^2 = B^2 = D^2 = -E$	$S^2 = \beta E$	$R^3 = S^2 = E$	$R^3 = E$	$R^3 = E$
$C^3 = I^2 = E$	$R^3 = T^2 = E$	$T^2 = \beta E$	$S^2 = T^2 = \beta E$	$S^2 = \beta E$
$BA = -AB$	$QP = \alpha PQ$	$QP = \alpha PQ$	$QP = \alpha PQ$	$QP = \alpha PQ$
$CA = BC$	$RP = QR$	$RP = QR$	$RP = QR$	$RP = QR$
$CB = ABC$	$RQ = PQR$	$RQ = PQR$	$RQ = PQR$	$RQ = PQR$
$DA = -BD$	$SP = \alpha QS$	$SP = \alpha QS$	$SP = \alpha QS$	$SP = \alpha QS$
$DB = -AD$	$SQ = \alpha PS$	$SQ = \alpha PS$	$SQ = \alpha PS$	$SQ = \alpha PS$
$DC = C^2D$	$SR = R^2S$	$SR = R^2S$	$SR = R^2S$	$SR = R^2S$
$IA = AI$	$TP = PT$	$TP = PT$	$TP = PT$	$TP = PT$
$IB = BI$	$TQ = QT$	$TQ = QT$	$TQ = QT$	$TQ = QT$
$IC = CI$	$TR = RT$	$TR = RT$	$TR = RT$	$TR = RT$
$ID = DI$	$TS = \alpha\beta ST$	$TS = \beta ST$	$TS = \beta ST$	$TS = \beta ST$
required gauge transformations	$P \to A$ $Q \to B$ $R \to C$ $S \to D$ $T \to I$	$P \to A$ $C \to B$ $R \to C$ $S \to \pm iD$ $T \to I$	$P \to A$ $Q \to B$ $R \to C$ $S \to \pm iD$ $T \to I$	$P \to A$ $Q \to B$ $R \leftarrow C$ $S \to \pm iD$ $T \to \pm iI$
required class of representations	$\alpha = -1$ $\beta = -1$	$\alpha = -1$ $\beta = +1$	$\alpha = -1$ $\beta = +1$	$\alpha = -1$ $\beta = -1$

The calculation of the double-valued representations then proceeds as in the preceding example of D_4' and identical sets of double-valued representations of O_h' are obtained from all four representation groups.

8.2 *Derivation of the single-valued, double-valued and protective representations of the space groups*

Koster (1957) reduced the problem of determining space group representations to that of determining the representations of $P(\boldsymbol{k})$, the space group of the $\boldsymbol{k}$-vector in reciprocal space. These are found from the representations of the quotient of $P(\boldsymbol{k})$ with the translation group. This is the point group $G_0(\boldsymbol{k})$. In general, however, the multiplication rules required for the representations of $G_0(\boldsymbol{k})$ will contain factor systems. Hurley (1966) noticed that Koster's results led to the conclusion that the vector representations of $G_0(\boldsymbol{k})$ were sufficient when dealing with points in the interior of the first Brillouin zone for non-symmorphic space groups and for all points in symmorphic space groups. Projective representations are, however, required for points on the surface or the outside of non-symmorphic space groups. Hurley (1966) showed how the space group representations could be derived from his tables of projective representations and we shall show that the space group representations are uniquely determined, irrespective of which set of projective representations, and hence which representation group, is chosen. However, where erroneous tables have been published these do indeed lead to incorrect space group representations. We shall also show that double-valued space group representations are easily obtainable from our tables of representation groups.

Our first example concerns the point R on the surface of the Brillouin zone of the space group $O_h^2(\equiv Pn3n)$. For this point, $G_0(\boldsymbol{k})$ is O_h and a suitable set of generators for this group can be derived from those given by Bradley & Cracknell (1972). These are, in Seitz notation,

$$A = \{C_{2x}\,|\,000\},\quad B = \{C_{2y}\,|\,000\},\quad C = \{C_{31}^{+}\,|\,000\},\quad D = \{C_{26}\,|\,000\};\quad I = \{S_2\,|\,\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\}$$

and direct application of Bradley & Cracknell's tables yields the relationship between these generators of $P(\boldsymbol{k})$. As in §8.1 these are compared with the generating relations for the matrices of the representation group to determine the relevant class of projective representations and also the phase factors by which their characters are to be modified:

$P(\boldsymbol{k})$	$\mathscr{R}_1(O_h)$	$\mathscr{R}_2(O_h)$	$\mathscr{R}_3(O_h)$	$\mathscr{R}_4(O_h)$
$A^2 = B^2 = E$	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = \alpha E$	$P^2 = Q^2 = T^2 = \alpha E$
$C^3 = E$	$S^2 = \beta E$	$R^3 = S^2 = E$	$R^3 = E$	$R^3 = E$
$D^2 = I^2 = E$	$R^3 = T^2 = E$	$T^2 = \beta E$	$S^2 = T^2 = \beta E$	$S^2 = \beta E$
$BA = AB$	$QP = \alpha PQ$	$QP = \alpha PQ$	$QP = \alpha PQ$	$QP = \alpha PQ$
$CA = BC$	$RP = QR$	$RP = QR$	$RP = QR$	$RP = QR$
$CB = ABC$	$RQ = PQR$	$RQ = PQR$	$RQ = PQR$	$RQ = PQR$
$DA = BD$	$SP = \alpha QS$	$SP = \alpha QS$	$SP = \alpha QS$	$SP = \alpha QS$
$DB = AD$	$SQ = \alpha PS$	$SQ = \alpha PS$	$SQ = \alpha PS$	$SQ = \alpha PS$
$DC = C^2D$	$SR = R^2S$	$SR = R^2S$	$SR = R^2S$	$SR = R^2S$
$IA = AI$	$TP = PT$	$TP = PT$	$TP = PT$	$TP = PT$
$IB = BI$	$TQ = QT$	$TQ = QT$	$TQ = QT$	$TQ = QT$
$IC = CI$	$TR = RT$	$TR = RT$	$TR = RT$	$TR = RT$
$ID = -DI$	$TS = \alpha\beta ST$	$TS = \beta ST$	$TS = \beta ST$	$TS = \beta ST$
required gauge transformation	$P \to A$ $Q \to B$ $R \to C$ $S \to \pm iD$ $T \to I$	$P \to A$ $Q \to B$ $R \to C$ $S \to D$ $T \to \pm iI$	$P \to A$ $Q \to B$ $R \to C$ $S \to \pm iD$ $T \to \pm iI$	$P \to A$ $Q \to B$ $R \to C$ $S \to \pm iD$ $T \to \pm iI$
required class of representations	$\alpha = +1$ $\beta = -1$	$\alpha = +1$ $\beta = -1$	$\alpha = +1$ $\beta = -1$	$\alpha = +1$ $\beta = -1$

Inspection of the appropriate classes of representations and division of the characters by the phase factors resulting from the gauge transformations confirms that the space group representations are unique.

A further example will usefully consider the point L in $O_h^8(\equiv Fd3c)$. The group $G_0(\boldsymbol{k})$ is D_{3h} and generating matrices for this are suitably chosen as $A = \{S_{61}^-|\frac{3}{4}\frac{3}{4}\frac{3}{4}\}$ and $B = \{C_{2b}|\frac{1}{4}\frac{1}{4}\frac{1}{4}\}$. The relations between these generators and those of the representation groups of D_{3h} are compared below:

$P(\boldsymbol{k})$	$\mathscr{R}_1(D_{3d})$	$\mathscr{R}_2(D_{3d})$	$\mathscr{R}_3(D_{3d})$
$A^6 = E$	$P^6 = \alpha E$	$P^6 = \alpha E$	$P^6 = E$
$B^2 = E$	$Q^2 = \alpha E$	$Q^2 = E$	$Q^2 = \alpha E$
$BA = -A^5B$	$QP = \alpha P^5Q$	$QP = \alpha P^5Q$	$QP = \alpha P^5Q$
required gauge transformations	$P \to \pm iA$ $Q \to \pm iB$	$P \to \pm iA$ $Q \to B$	$P \to A$ $Q \to \pm iB$
required class of representations	$\alpha = -1$	$\alpha = -1$	$\alpha = -1$

In all three cases, and for all choices of $\pm$ signs in the phase factors, the same space group representations result.

The final example concerns the double-valued representations of the point R of O_h^2 discussed in the first example. The relations between the generating matrices for $P(\boldsymbol{k})$ differ from those for the single-valued representations only in the signs of A^2, B^2, BA, DA and DB. The appropriate gauge transformations and choices of representations are therefore

$$\mathscr{R}_1(O_h): \quad P \to A, \quad Q \to B, \quad R \to C, \quad S \to D, \quad T \to I; \quad \alpha = -1, \quad \beta = -1$$

$$\mathscr{R}_2(O_h): \quad P \to A, \quad Q \to B, \quad R \to C, \quad S \to D, \quad T \to \pm iI; \quad \alpha = -1, \quad \beta = -1$$

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$$\mathscr{R}_3(O_h): \quad P \to A, \quad Q \to B, \quad R \to C, \quad S \to \pm \mathrm{i} D, \quad T \to I; \qquad \alpha = +1, \quad \beta = -1$$

$$\mathscr{R}_4(O_h): \quad P \to A, \quad Q \to B, \quad R \to C, \quad S \to \pm \mathrm{i} D, \quad T \to \pm \mathrm{i} I; \quad \alpha = -1, \quad \beta = -1$$

The double-valued space group representations so produced are again unique, irrespective of the choice of representation group.

The projective representations of the space groups, recently discussed by Bradley & Backhouse (1970, 1972) and Backhouse (1970, 1971) could also be straightforwardly derived from our representation group tables. The advantage of these is that they allow one to construct the equivalent, but different, sets of projective representations and hence give greater flexibility for ascending and descending in symmetry.

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Table 3 (cont.)

	$1c_1$	$1c_2$	$1c_2$	$1c_2$	$1c_2$	$1c_2$	$1c_2$	$1c_2$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_4$	$4c_2$	$4c_3$	64 elements: $P^4=Q^4=R^4=E$; $QP=P^3Q$; $RQ=Q^3R$; $PR=R^3P$		
$\mathscr{R}(D_{2h})$	E	P^2	Q^2	R^2	P^2Q^2	P^2R^2	Q^2R^2	$P^2Q^2R^2$	PQ^2, P^3Q^2, $PQ^2R^2, P^3Q^2R^2$	QR^2, Q^3R^2, $P^2QR^2, P^2Q^3R^2$	P^2R, P^2R^3, $P^2Q^2R, P^2Q^2R^3$	P, P^3, PR^2, P^3R^2	Q, Q^3, P^2Q, P^2Q^3	R, R^3, Q^2R, Q^2R^3	PQ, P^3Q, PQ^2R^2, $P^3Q^3R^2$	PQ^3, P^3Q^3, PQR^2, P^3QR^2	PR, PR^3, P^3Q^2R, $P^3Q^2R^3$	P^3R, P^3R^3, PQ^2R, PQ^2R^3	QR, Q^3R, P^2QR^3, $P^2Q^3R^3$	QR^3, Q^3R^3, P^2QR, P^2Q^3R	PQR, P^3QR^3, P^3Q^3R, PQ^3R^3	PQR^3, PQ^3R, P^3QR, $P^3Q^3R^3$	α	β	γ
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B_{1g}	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	−1	−1	1	1	1	1			
B_{2g}	1	1	1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1			
B_{3g}	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	1	1	1	−1	−1	−1	−1	1	1			
A_u	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1			
B_{1u}	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1			
B_{2u}	1	1	1	1	1	1	1	1	1	−1	1	1	−1	1	−1	−1	1	1	−1	−1	−1	−1			
B_{3u}	1	1	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	−1	−1	−1	−1	−1	−1			
E'_α	2	2	−2	2	−2	2	−2	−2	2	0	0	−2	0	0	0	0	0	0	0	0	0	0	−1	1	1
E''_α	2	2	−2	2	−2	2	−2	−2	−2	0	0	2	0	0	0	0	0	0	0	0	0	0			
E'_β	2	2	2	−2	2	−2	−2	−2	0	2	0	0	−2	0	0	0	0	0	0	0	0	0	1	−1	1
E''_β	2	2	2	−2	2	−2	−2	−2	0	−2	0	0	2	0	0	0	0	0	0	0	0	0			
E'_γ	2	−2	2	2	−2	−2	2	−2	0	0	2	0	0	−2	0	0	0	0	0	0	0	0	1	1	−1
E''_γ	2	−2	2	2	−2	−2	2	−2	0	0	−2	0	0	2	0	0	0	0	0	0	0	0			
$G_{\alpha\beta}$ $G^+_{\alpha\beta}$	2	2	−2	−2	−2	−2	2	2	0	0	0	0	0	0	2i	−2i	0	0	0	0	0	0	−1	−1	1
$G^-_{\alpha\beta}$	2	2	−2	−2	−2	−2	2	2	0	0	0	0	0	0	−2i	2i	0	0	0	0	0	0			
$G_{\alpha\gamma}$ $G^+_{\alpha\gamma}$	2	−2	−2	2	2	−2	−2	2	0	0	0	0	0	0	0	0	2i	−2i	0	0	0	0	−1	1	−1
$G^-_{\alpha\gamma}$	2	−2	−2	2	2	−2	−2	2	0	0	0	0	0	0	0	0	−2i	2i	0	0	0	0			
$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	2	−2	2	−2	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	2i	−2i	0	0	1	−1	−1
$G^-_{\beta\gamma}$	2	−2	2	−2	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	−2i	2i	0	0			
$E'_{\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	0	0	0	0	0	0	0	0	0	0	0	0	2	−2	−1	−1	−1
$E''_{\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	0	0	0	0	0	0	0	0	0	0	0	0	−2	2			
D_{2h}	E											A	B	C	AB		AC		BC		ABC		$A^2=B^2=C^2=E$; $AB=BA$; $AC=CA$; $BC=CB$		

TABLE 3 (cont.)

	$1e_1$	$1e_2$	$1e_2$	$1e_2$	$1e_2$	$1e_2$	$1e_2$	$1e_2$	$1 \leqslant p \leqslant n-1$ $2e_{2n}$	$1 \leqslant p \leqslant n-1$ $2e_{2n}$	$1 \leqslant p \leqslant n-1$ $2e_{2n}$	$1 \leqslant p \leqslant n$ $4e_{4n/\mathrm{hcf}(4n,2p-1)}$	$1 \leqslant p \leqslant n-1$ $2e_{2n/\mathrm{hcf}(2n,p)}$	$1 \leqslant p \leqslant n$ $4e_{4n/\mathrm{hcf}(4n,2p-1)}$	$1 \leqslant p \leqslant 2n$ $4e_{4n/\mathrm{hcf}(4n,2p-1)}$	$0 \leqslant p \leqslant 2n-1$ $4e_{4n/\mathrm{hcf}(2n,p)}$	$4ne_4$ $0 \leqslant q \leqslant 2n-1$	$4ne_4$ $0 \leqslant q \leqslant 2n-1$	$4ne_4$ $1 \leqslant q \leqslant 2n$	$4ne_4$ $1 \leqslant q \leqslant 2n$	$4ne_4$ $0 \leqslant q \leqslant n-1$	$4ne_4$ $0 \leqslant q \leqslant n-1$	$4ne_2$ $1 \leqslant q \leqslant n$	$4ne_2$ $1 \leqslant q \leqslant n$	$64n$ elements
$\mathscr{R}_1(D_{2nh})$	E	P^{2n}	Q^2	R^2	Q^2R^2	$P^{2n}Q^2$	$P^{2n}R^2$	$P^{2n}Q^2R^2$	$P^{4n-2p}Q^2$ $P^{2p}Q^2$	$P^{4n-2p}R^2$ $P^{2p}R^2$	$P^{4n-2p}R^2$ $P^{2p}Q^2R^2$	$P^{4n+1-2p}R^2$ $P^{2p-1}R^2$ $P^{4n+1-2p}$ P^{2p-1}	P^{4n-2p} P^{2p}	$P^{4n+1-2p}Q^2R^2$ $P^{2p-1}Q^2R^2$ $P^{4n+1-2p}Q^2$ $P^{2p-1}Q^2$	$P^{4n+1-2p}Q^2R^3$ $P^{4n+1-2p}Q^2R$ $P^{2p-1}R^3$ $P^{2p-1}R$	$P^{4n-2p}Q^2R^3$ $P^{4n-2p}Q^2R$ $P^{2p}R^3$ $P^{2p}R$	$P^{2q}Q^3$ $P^{2q}Q$	$P^{2q}Q^3R^2$ $P^{2q}QR^2$	$P^{2q-1}Q^3R^2$ $P^{2q-1}Q$	$P^{2q-1}QR^2$ $P^{2q-1}Q^3$	$P^{4q+2}Q^3R$ $P^{4q+2}QR$ $P^{4q}Q^3R^3$ $P^{4q}QR^3$	$P^{4q+2}Q^3R^3$ $P^{4q+2}QR^3$ $P^{4q}Q^3R$ $P^{4q}QR$	$P^{4q-3}Q^3R^3$ $P^{4q-3}QR$ $P^{4q-1}QR^3$ $P^{4q-1}Q^3R$	$P^{4q-3}QR^3$ $P^{4q-3}Q^3R$ $P^{4q-1}Q^3R^3$ $P^{4q-1}QR$	$P^{4n}=Q^4=R^4=E$ $QP=P^{4n-1}Q$; $RQ=Q^3R$ $PR=R^3P$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\alpha=+1;\beta=+1;\gamma=+1$
A_{2g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	
B_{1g}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	$(-1)^{n-1}$	$(-1)^n$	1	1	−1	−1	1	1	−1	−1	
B_{2g}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	$(-1)^{n-1}$	$(-1)^n$	−1	−1	1	1	−1	−1	1	1	
A_{1u}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	
A_{2u}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	−1	−1	−1	−1	
B_{1u}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	$(-1)^n$	$(-1)^{n-1}$	−1	−1	1	1	1	1	−1	−1	
B_{2u}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	$(-1)^n$	$(-1)^{n-1}$	1	1	−1	−1	−1	−1	1	1	
$\leqslant l \leqslant n-1$; E_{lg}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$(-1)^l 2\cos\{l(2p-1)\pi/n\}$	$(-1)^l 2\cos\{2lp\pi/n\}$	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant n-1$; E_{lu}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$(-1)^{l+1} 2\cos\{l(2p-1)\pi/n\}$	$(-1)^{l+1} 2\cos\{2lp\pi/n\}$	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 2n$; $E_{l\alpha}$	2	−2	2	2	2	−2	−2	−2	$2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$(-1)^l 2\cos\{(2l-1)(2p-1)\pi/2n\}$	$(-1)^l 2\cos\{(2l-1)p\pi/n\}$	0	0	0	0	0	0	0	0	$\alpha=-1;\beta=+1;\gamma=+1$
$E_{1\beta}$	2	2	−2	2	−2	−2	2	−2	−2	2	−2	2	2	−2	0	0	0	0	0	0	0	0	0	0	$\alpha=+1;\beta=-1;\gamma=+1$
$E_{2\beta}$	2	2	−2	2	−2	−2	2	−2	−2	2	−2	−2	2	2	0	0	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant n-1$; $G_{l\beta}$ $\{G^+_{l\beta}$	2	2	−2	2	−2	−2	2	−2	$-2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\{l(2p-1)\pi/n\}$	$2\mathrm{i}\sin\{l(2p-1)\pi/n\}$	$2\mathrm{i}\sin\{2lp\pi/n\}$	0	0	0	0	0	0	0	0	
$G^-_{l\beta}$	2	2	−2	2	−2	−2	2	−2	$-2\cos\{2lp\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\{2lp\pi/n\}$	$2\cos\{l(2p-1)\pi/n\}$	$2\cos\{2lp\pi/n\}$	$-2\cos\{l(2p-1)\pi/n\}$	$-2\mathrm{i}\sin\{l(2p-1)\pi/n\}$	$-2\mathrm{i}\sin\{2lp\pi/n\}$	0	0	0	0	0	0	0	0	
$E_{1\gamma}$	2	2	2	−2	−2	2	−2	−2	2	−2	−2	0	2	0	0	0	2	−2	0	0	0	0	0	0	$\alpha=+1;\beta=+1;\gamma=-1$
$E_{2\gamma}$	2	2	2	−2	−2	2	−2	−2	2	−2	−2	0	2	0	0	0	−2	2	0	0	0	0	0	0	
even; G_γ $\{G^+_\gamma$	2	2	2	−2	−2	2	−2	−2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	2i	−2i	0	0	
G^-_γ	2	2	2	−2	−2	2	−2	−2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	−2i	2i	0	0	
$\leqslant l \leqslant \frac{1}{2}(n-1)$; $G_{l\gamma}$	4	4	4	−4	−4	4	−4	−4	$4\cos\{2lp\pi/n\}$	$-4\cos\{2lp\pi/n\}$	$-4\cos\{2lp\pi/n\}$	0	$4\cos\{2pl\pi/n\}$	0	0	0	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant n$; $G_{l\alpha\beta}$ $\{G^+_{l\alpha\beta}$	2	−2	−2	2	−2	2	−2	2	$-2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)p\pi/n\}$	$-2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)p\pi/n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)(2p-1)\pi/2n\}$	$2\mathrm{i}\sin\{(2l-1)p\pi/n\}$	0	0	0	0	0	0	0	0	$\alpha=-1;\beta=-1;\gamma=+1$
$G^-_{l\alpha\beta}$	2	−2	−2	2	−2	2	−2	2	$-2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)p\pi/n\}$	$-2\cos\{(2l-1)p\pi/n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)p\pi/n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)(2p-1)\pi/2n\}$	$-2\mathrm{i}\sin\{(2l-1)p\pi/n\}$	0	0	0	0	0	0	0	0	
odd; $G_{\alpha\gamma}$ $\{G^+_{\alpha\gamma}$	2	−2	2	−2	−2	−2	2	2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	2i	−2i	0	0	$\alpha=-1;\beta=+1;\gamma=-1$
$G^-_{\alpha\gamma}$	2	−2	2	−2	−2	−2	2	2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	−2i	2i	0	0	
$\leqslant l \leqslant \frac{1}{2}n$; $G_{l\alpha\gamma}$	4	−4	4	−4	−4	−4	4	4	$4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	0	0	0	0	0	0	0	0	0	0	0	
$G_{\beta\gamma}$ $\{G^+_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	−2	−2	2	0	2	0	0	0	0	0	2i	−2i	0	0	0	0	$\alpha=+1;\beta=-1;\gamma=-1$
$G^-_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	−2	−2	2	0	2	0	0	0	0	0	−2i	2i	0	0	0	0	
even; $E_{1\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	2	−2	
even; $E_{2\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	−2	2	
$\leqslant l \leqslant \frac{1}{2}(n-1)$; $G_{l\beta\gamma}$	4	4	−4	−4	4	−4	−4	4	$-4\cos\{2lp\pi/n\}$	$-4\cos\{2lp\pi/n\}$	$4\cos\{2lp\pi/n\}$	0	$4\cos\{2lp\pi/n\}$	0	0	0	0	0	0	0	0	0	0	0	
odd; $E_{1\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	2	−2	$\alpha=-1;\beta=-1;\gamma=-1$
odd; $E_{2\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	−2	2	
$\leqslant l \leqslant \frac{1}{2}n$; $G_{l\alpha\beta\gamma}$	4	−4	−4	−4	4	4	4	−4	$-4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	$4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	0	0	0	0	0	0	0	0	0	0	0	
D_{2nh}	E											A^{2p-1}	A^{2p}		$A^{2p-1}C$	$A^{2p}C$	$A^{2q}B$ $0 \leqslant q \leqslant n-1$		$A^{2q-1}B$ $1 \leqslant q \leqslant n$		$A^{4q}BC$ $0 \leqslant q \leqslant \frac{1}{2}(n-1)$	$A^{4q-2}BC$ $1 \leqslant q \leqslant \frac{1}{2}n$	$A^{4q-3}BC$ $1 \leqslant q \leqslant \frac{1}{2}(n+1)$	$A^{4q-1}BC$ $1 \leqslant q \leqslant \frac{1}{2}n$	$A^{2n}=B^2=C^2=E$ $BA=A^{2n-1}B$ $CA=AC$; $CB=BC$

TABLE 3 (*cont.*)

$\mathscr{R}_2(D_{4nh})$	$1c_1$ E	$1c_2$ P^{4n}	$1c_2$ Q^2	$1c_2$ R^2	$1c_2$ Q^2R^2	$1c_2$ $P^{4n}Q^2$	$1c_2$ $P^{4n}R^2$	$1c_2$ $P^{4n}Q^2R^2$
A_{1g}	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	1	1	1	1
B_{1g}	1	1	1	1	1	1	1	1
B_{2g}	1	1	1	1	1	1	1	1
A_{1u}	1	1	1	1	1	1	1	1
A_{2u}	1	1	1	1	1	1	1	1
B_{1u}	1	1	1	1	1	1	1	1
B_{2u}	1	1	1	1	1	1	1	1
$1 \leqslant l \leqslant 2n-1$; E_{lg}	2	2	2	2	2	2	2	2
$1 \leqslant l \leqslant 2n-1$; E_{lu}	2	2	2	2	2	2	2	2
$1 \leqslant l \leqslant 2n$; $G_{l\alpha}$ $G^+_{l\alpha}$	2	−2	2	2	2	−2	−2	−2
$G^-_{l\alpha}$	2	−2	2	2	2	−2	−2	−2
$E_{1\beta}$	2	2	−2	2	−2	−2	2	−2
$E_{2\beta}$	2	2	−2	2	−2	−2	2	−2
$\leqslant l \leqslant 2n-1$; $G_{l\beta}$ $G^+_{l\beta}$	2	2	−2	2	−2	−2	2	−2
$G^-_{l\beta}$	2	2	−2	2	−2	−2	2	−2
$E_{1\gamma}$	2	2	2	−2	−2	2	−2	−2
$E_{2\gamma}$	2	2	2	−2	−2	2	−2	−2
G_γ G^+_γ	2	2	2	−2	−2	2	−2	−2
G^-_γ	2	2	2	−2	−2	2	−2	−2
$1 \leqslant l \leqslant n-1$; $G_{l\gamma}$	4	4	4	−4	−4	4	−4	−4
$1 \leqslant l \leqslant 2n$; $G_{l\alpha\beta}$ $G^+_{l\alpha\beta}$	2	−2	−2	2	−2	2	−2	2
$G^-_{l\alpha\beta}$	2	−2	−2	2	−2	2	−2	2
$1 \leqslant l \leqslant n$; $G_{l\alpha\gamma}$	4	−4	4	−4	−4	−4	4	4
$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2
$G^-_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2
$E_{1\beta\gamma}$	2	2	−2	−2	2	−2	−2	2
$E_{2\beta\gamma}$	2	2	−2	−2	2	−2	−2	2
$1 \leqslant l \leqslant n-1$; $G_{l\beta\gamma}$	4	4	−4	−4	4	−4	−4	4
$1 \leqslant l \leqslant n$; $G_{l\alpha\beta\gamma}$	4	−4	−4	−4	4	4	4	−4
D_{4nh}	E							

$\mathscr{R}_2(D_{4nh})$	$1 \leqslant p \leqslant 2n-1$ $2c_{4n/\mathrm{hcf}(4n,p)}$ $P^{8n-2p}Q^2$ $P^{2p}Q^2$	$1 \leqslant p \leqslant 2n-1$ $2c_{4n/\mathrm{hcf}(4n,p)}$ $P^{8n-2p}R^2$ $P^{2p}R^2$	$1 \leqslant p \leqslant 2n-1$ $2c_{4n/\mathrm{hcf}(4n,p)}$ $P^{8n-2p}Q^2R^2$ $P^{2p}Q^2R^2$	$1 \leqslant p \leqslant 2n$ $4c_{8n/\mathrm{hcf}(8n,4p-3)}$ $P^{4n+3-4p}R^2$ $P^{4p-3}R^2$ $P^{4n+3-4p}$ P^{4p-3}	$1 \leqslant p \leqslant 2n-1$ $2c_{8n/\mathrm{hcf}(4n,p)}$ P^{8n-2p} P^{2p}	$1 \leqslant p \leqslant 4n$ $4c_{8n/\mathrm{hcf}(2p-1,n)}$ $P^{4n+1-2p}Q^2R^3$ $P^{4n+1-2p}Q^2R$ $P^{2p-1}R^3$ $P^{2p-1}R$	$0 \leqslant p \leqslant 4n-1$ $4c_{4/\mathrm{hcf}(n,p)}$ $P^{8n-2p}Q^2R^3$ $P^{8n-2p}Q^2R$ $P^{2p}R^3$ $P^{2p}R$	$1 \leqslant p \leqslant 2n$ $4c_{8n/\mathrm{hcf}(8n,4p-3)}$ $P^{4n+3-4p}Q^2R^2$ $P^{4p-3}Q^2R^2$ $P^{4n+3-4p}Q^2$ $P^{4p-3}Q^2$
A_{1g}	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	1	1	1	1
B_{1g}	1	1	1	−1	1	−1	$(-1)^p$	−1
B_{2g}	1	1	1	−1	1	−1	$(-1)^p$	−1
A_{1u}	1	1	1	1	1	−1	−1	1
A_{2u}	1	1	1	1	1	−1	−1	1
B_{1u}	1	1	1	−1	1	1	$(-1)^{p+1}$	−1
B_{2u}	1	1	1	−1	1	1	$(-1)^{p+1}$	−1
$1 \leqslant l \leqslant 2n-1$; E_{lg}	$2\cos(lp\pi/n)$	$2\cos(lp\pi/n)$	$2\cos(lp\pi/n)$	$2\cos\{l(4p-3)\pi/2n\}$	$2\cos(lp\pi/n)$	$2\cos\{l(2p-1)\pi/2n\}$	$2\cos(lp\pi/n)$	$2\cos\{l(4p-3)\pi/2n\}$
$1 \leqslant l \leqslant 2n-1$; E_{lu}	$2\cos(lp\pi/n)$	$2\cos(lp\pi/n)$	$2\cos(lp\pi/n)$	$2\cos\{l(4p-3)\pi/2n\}$	$2\cos(lp\pi/n)$	$-2\cos\{l(2p-1)\pi/2n\}$	$-2\cos(lp\pi/n)$	$2\cos\{l(4p-3)\pi/2n\}$
$1 \leqslant l \leqslant 2n$; $G_{l\alpha}$ $G^+_{l\alpha}$	$2\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$2i\sin\{(2l+1)(2p-1)\pi/4n\}$	$(-1)^{l+1}2\cos\{(2l-1)\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$
$G^-_{l\alpha}$	$2\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l+1)(2p-1)\pi/4n\}$	$(-1)^{l+1}2\cos\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$
$E_{1\beta}$	−2	2	−2	2	2	0	0	−2
$E_{2\beta}$	−2	2	−2	−2	2	0	0	2
$\leqslant l \leqslant 2n-1$; $G_{l\beta}$ $G^+_{l\beta}$	$2(-1)^{p+1}$	$2(-1)^p$	$2(-1)^{p+1}$	0	$2(-1)^p$	$2i\sin\{(2l-1)(2p-1)\pi/2n\}$	$2i\sin\{(2l-1)p\pi/n\}$	0
$G^-_{l\beta}$	$2(-1)^{p+1}$	$2(-1)^p$	$2(-1)^{p+1}$	0	$2(-1)^p$	$-2i\sin\{(2l-1)(2p-1)\pi/2n\}$	$-2i\sin\{(2l-1)p\pi/n\}$	0
$E_{1\gamma}$	2	−2	−2	0	2	0	0	0
$E_{2\gamma}$	2	−2	−2	0	2	0	0	0
G_γ G^+_γ	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0
G^-_γ	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0
$1 \leqslant l \leqslant n-1$; $G_{l\gamma}$	$4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	0	0	0
$1 \leqslant l \leqslant 2n$; $G_{l\alpha\beta}$ $G^+_{l\alpha\beta}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)(2p-1)\pi/4n\}$	$2i\sin\{(2l-1)p\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$
$G^-_{l\alpha\beta}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$2\cos\{(2l-1)(2p-1)\pi/2n\}$	$-2\cos\{(2l-1)(2p-1)\pi/2n\}$	$-2i\sin\{(2l-1)(4p-3)\pi/4n\}$	$2\cos(2l-1)(2p-1)\pi/2n)$	$2\cos\{(2l-1)(2p-1)\pi/4n\}$	$-2i\sin\{(2l-1)p\pi/2n\}$	$2i\sin\{(2l-1)(4p-3)\pi/4n\}$
$1 \leqslant l \leqslant n$; $G_{l\alpha\gamma}$	$4\cos\{(2l-1)p\pi/2n\}$	$-4\cos\{(2l-1)p\pi/2n\}$	$-4\cos\{(2l-1)p\pi/2n\}$	0	$4\cos\{(2l-1)p\pi/2n\}$	0	0	0
$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	−2	−2	2	0	2	0	0	0
$G^-_{\beta\gamma}$	−2	−2	2	0	2	0	0	0
$E_{1\beta\gamma}$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0
$E_{2\beta\gamma}$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	$2(-1)^p$	0	$2(-1)^p$	0	0	0
$1 \leqslant l \leqslant n-1$; $G_{l\beta\gamma}$	$-4\cos\{(2l-1)p\pi/n\}$	$-4\cos\{(2l-1)p\pi/n\}$	$4\cos\{(2l-1)p\pi/n\}$	0	$4\cos\{(2l-1)p\pi/n\}$	0	0	0
$1 \leqslant l \leqslant n$; $G_{l\alpha\beta\gamma}$	$-4\cos\{(2l-1)p\pi/2n\}$	$-4\cos\{(2l-1)p\pi/2n\}$	$4\cos\{(2l-1)p\pi/2n\}$	0	$4\cos\{(2l-1)p\pi/2n\}$	0	0	0
D_{4nh}				A^{2p-1} $A^{4n+1-2p}$ $1 \leqslant p \leqslant n$	A^{2p}	$A^{2p-1}C$ $1 \leqslant p \leqslant 2n$	$A^{2p}C$ $0 \leqslant p \leqslant 2n-1$	

$\mathscr{R}_2(D_{4nh})$	$8nc_4$ $0 \leqslant q \leqslant 4n-1$ $P^{2q}Q^3$ $P^{2q}Q$	$8nc_4$ $0 \leqslant q \leqslant 4n-1$ $P^{2q}Q^3R^2$ $P^{2q}QR^2$	$8nc_4$ $1 \leqslant q \leqslant 4n$ $P^{2q-1}Q^3R^2$ $P^{2q-1}Q$	$8nc_4$ $1 \leqslant q \leqslant 4n$ $P^{2q-1}QR^2$ $P^{2q-1}Q^3$	$8nc_4$ $0 \leqslant q \leqslant n-1$ $P^{4q-2}Q^3R^3$ $P^{4q-4}Q^3R$ $P^{4q-2}QR^3$ $P^{4q-4}QR$	$8nc_4$ $0 \leqslant q \leqslant n-1$ $P^{4q-2}Q^3R$ $P^{4q-4}Q^3R^3$ $P^{4q-2}QR$ $P^{4q-4}QR^3$	$8nc_4$ $0 \leqslant q \leqslant n-1$ $P^{4q-1}Q^3R^3$ $P^{4q-3}Q^3R$ $P^{4q-1}QR^3$ $P^{4q-3}QR$	$8nc_4$ $0 \leqslant q \leqslant n-1$ $P^{4q-1}Q^3R$ $P^{4q-3}Q^3R^3$ $P^{4q-1}QR$ $P^{4q-3}QR^3$	$128n$ elements $P^{8n} = Q^4 = R^4 = E$ $QP = P^{4n-1}Q$ $RP = PR^3$ $RQ = Q^3R$
A_{1g}	1	1	1	1	1	1	1	1	$\alpha = +1$; $\beta = +1$; $\gamma = +1$
A_{2g}	−1	−1	−1	−1	−1	−1	−1	−1	
B_{1g}	1	1	−1	−1	1	1	−1	−1	
B_{2g}	−1	−1	1	1	−1	−1	1	1	
A_{1u}	−1	−1	−1	−1	1	1	1	1	
A_{2u}	1	1	1	1	−1	−1	−1	−1	
B_{1u}	−1	−1	1	1	1	1	−1	−1	
B_{2u}	1	1	−1	−1	−1	−1	1	1	
$1 \leqslant l \leqslant 2n-1$; E_{lg}	0	0	0	0	0	0	0	0	
$1 \leqslant l \leqslant 2n-1$; E_{lu}	0	0	0	0	0	0	0	0	
$1 \leqslant l \leqslant 2n$; $G_{l\alpha}$ $G^+_{l\alpha}$	0	0	0	0	0	0	0	0	$\alpha = -1$; $\beta = +1$; $\gamma = +1$
$G^-_{l\alpha}$	0	0	0	0	0	0	0	0	
$E_{1\beta}$	0	0	0	0	0	0	0	0	$\alpha = +1$; $\beta = -1$; $\gamma = +1$
$E_{2\beta}$	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 2n-1$; $G_{l\beta}$ $G^+_{l\beta}$	0	0	0	0	0	0	0	0	
$G^-_{l\beta}$	0	0	0	0	0	0	0	0	
$E_{1\gamma}$	2	−2	0	0	0	0	0	0	$\alpha = +1$; $\beta = +1$; $\gamma = -1$
$E_{2\gamma}$	−2	2	0	0	0	0	0	0	
G_γ G^+_γ	0	0	0	0	2i	−2i	0	0	
G^-_γ	0	0	0	0	−2i	2i	0	0	
$1 \leqslant l \leqslant n-1$; $G_{l\gamma}$	0	0	0	0	0	0	0	0	
$1 \leqslant l \leqslant 2n$; $G_{l\alpha\beta}$ $G^+_{l\alpha\beta}$	0	0	0	0	0	0	0	0	$\alpha = -1$; $\beta = -1$; $\gamma = +1$
$G^-_{l\alpha\beta}$	0	0	0	0	0	0	0	0	
$1 \leqslant l \leqslant n$; $G_{l\alpha\gamma}$	0	0	0	0	0	0	0	0	$\alpha = -1$; $\beta = +1$; $\gamma = -1$
$G_{\beta\gamma}$ $G^+_{\beta\gamma}$	0	0	2i	−2i	0	0	0	0	$\alpha = +1$; $\beta = -1$; $\gamma = -1$
$G^-_{\beta\gamma}$	0	0	−2i	2i	0	0	0	0	
$E_{1\beta\gamma}$	0	0	0	0	0	0	2	−2	
$E_{2\beta\gamma}$	0	0	0	0	0	0	−2	2	
$1 \leqslant l \leqslant n-1$; $G_{l\beta\gamma}$	0	0	0	0	0	0	0	0	
$1 \leqslant l \leqslant n$; $G_{l\alpha\beta\gamma}$	0	0	0	0	0	0	0	0	$\alpha = -1$; $\beta = -1$; $\gamma = -1$
D_{4nh}	$A^{2q}B$ $0 \leqslant q \leqslant 2n-1$		$A^{2q-1}B$ $1 \leqslant q \leqslant 2n$		$A^{4q}BC$ $0 \leqslant q \leqslant n-1$	$A^{4q+2}BC$ $0 \leqslant q \leqslant n-1$	$A^{4q+1}BC$ $0 \leqslant q \leqslant n-1$	$A^{4q+3}BC$ $0 \leqslant q \leqslant n-1$	$A^{4n} = B^2 = C^2 = E$ $BA = A^{2n-1}B$ $CA = AC$; $CB = BC$

TABLE 3 (*cont.*)

$\mathscr{R}_2(D_{(4n-2)h})$	$1\epsilon_1$ E	$1\epsilon_2$ P^{4n-2}	$1\epsilon_2$ Q^2	$1\epsilon_2$ R^2	$1\epsilon_2$ Q^2R^2	$1\epsilon_2$ $P^{4n-2}Q^2$	$1\epsilon_2$ $P^{4n-2}R^2$	$1\epsilon_2$ $P^{4n-2}Q^2R^2$	$1 \leqslant p \leqslant 2n-2$ $2\epsilon_{4n-2}$ $P^{8n-4-2p}Q^2$ $P^{2p}Q^2$	$1 \leqslant p \leqslant 2n-2$ $2\epsilon_{4n-2}$ $P^{8n-4-2p}R^2$ $P^{2p}R^2$	$1 \leqslant p \leqslant 2n-2$ $2\epsilon_{4n-2}$ $P^{8n-4-2p}Q^2R^2$ $P^{2p}Q^2R^2$	$1 \leqslant p \leqslant 2n-1$ $4\epsilon_{(8n-4)/\mathrm{hcf}(4n-2,2p-1)}$ $P^{4n-3+2p}Q^2$ $P^{4n-1-2p}Q^2$ $P^{8n-3-2p}$ P^{2p-1}	$1 \leqslant p \leqslant 2n-2$ $2\epsilon_{(4n-2)/\mathrm{hcf}(4n-2,p)}$ $P^{8n-4-2p}$ P^{2p}	$1 \leqslant p \leqslant 4n-2$ $4\epsilon_{(4n-2)/\mathrm{hcf}(4n-2,2p-1)}$ $P^{4n-1-2p}Q^3R^3$ $P^{4n-3+2p}Q^3R$ $P^{8n-3-2p}QR^3$ $P^{2p-1}QR$	$0 \leqslant p \leqslant 4n-3$ $4\epsilon_{(8n-4)/\mathrm{hcf}(2n-1,p)}$ $P^{4n-2-2p}Q^3R^3$ $P^{4n-2+2p}Q^3R$ $P^{8n-4-2p}QR^3$ $P^{2p}QR$	$1 \leqslant p \leqslant 2n-1$ $4\epsilon_{(8n-4)/\mathrm{hcf}(4n-2,2p-1)}$ $P^{4n-3+2p}Q^2R^3$ $P^{4n-1-2p}Q^2R^3$ $P^{8n-3-2p}R^3$ $P^{2p-1}R^3$	$(8n-4)\epsilon_4$ $0 \leqslant q \leqslant 4n-3$ $P^{2q}R^3$ $P^{2q}R$	$(8n-4)\epsilon_4$ $0 \leqslant q \leqslant 4n-3$ $P^{2}Q^2R^3$ $P^{2q}Q^2R$	$(8n-4)\epsilon_4$ $1 \leqslant q \leqslant 4n-2$ $P^{8n-2q+1}Q^2R^3$ $P^{8n-2q+1}R$	$(8n-4)\epsilon_4$ $1 \leqslant q \leqslant 4n-2$ $P^{8n-2q+1}Q^2R$ $P^{8n-2q+1}R^3$	$(8n-4)\epsilon_4$ $0 \leqslant q \leqslant 2n-2$ $P^{4n-4q}Q^3R^2$ $P^{4n-4q}QR^2$ $P^{4n-4q}Q^3$ $P^{4n-4q}Q$	$(8n-4)\epsilon_4$ $0 \leqslant q \leqslant 2n-2$ $P^{4n-4q+2}Q^3R^2$ $P^{4n-4q+2}QR^2$ $P^{4n-4q+2}Q^3$ $P^{4n-4q+2}Q$	$(8n-4)\epsilon_4$ $1 \leqslant q \leqslant 2n-1$ $P^{8n-1-4q}Q^3R^2$ $P^{8n-1-4q}QR^2$ $P^{8n-3-4q}Q^3$ $P^{8n-3-4q}Q$	$(8n-4)\epsilon_4$ $1 \leqslant q \leqslant 2n-1$ $P^{8n-3-4q}Q^3R^2$ $P^{8n-3-4q}QR^2$ $P^{8n-1-4q}Q^3$ $P^{8n-1-4q}Q$	$(128n-64)$ elements $P^{8n-4}=Q^4=R^4=E$ $QP=P^{8n-5}Q^3$ $RP=P^{8n-5}R$ $RQ=QR^3$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
A_{2g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	
B_{1g}	1	1	1	1	1	1	1	2	1	1	1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1	
B_{2g}	1	1	1	1	1	1	1	1	1	1	1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	
A_{1u}	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	$\alpha=+1;\beta=+1;\gamma=+1$
A_{2u}	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	
B_{1u}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	−1	−1	1	1	1	1	−1	−1	
B_{2u}	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	1	1	−1	−1	−1	−1	1	1	
$\leqslant l \leqslant 2n-2;$ E_{1g}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 2n-2;$ E_{1u}	2	2	2	2	2	2	2	2	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{l(2p-1)\pi/(n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	
$E_{1\alpha}$	2	−2	2	2	2	−2	−2	−2	$2(-1)^p$	$2(-1)^p$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	2	−2	0	0	
$E_{2\alpha}$	2	−2	2	2	2	−2	−2	−2	$2(-1)^p$	$2(-1)^p$	$2(-1)^p$	0	$2(-1)^p$	0	0	0	0	0	0	0	−2	2	0	0	$\alpha=-1;\beta=+1;\gamma=+1$
$\leqslant l \leqslant n-1;$ $G_{1\alpha}$	4	−4	4	4	4	−4	−4	−4	$4\cos\{(2l-1)p\pi/(2n-1)\}$	$4\cos(2l-1)p\pi/(2n-1)\}$	$4\cos\{(2l-1)p\pi/(2n-1)\}$	0	$4\cos\{(2l-1)p\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	0	0	0	
$E_{1\beta}$	2	2	−2	2	−2	−2	2	−2	−2	2	−2	0	2	0	0	0	2	−2	0	0	0	0	0	0	
$E_{2\beta}$	2	2	−2	2	−2	−2	2	−2	−2	2	−2	0	2	0	0	0	−2	2	0	0	0	0	0	0	$\alpha=+1;\beta=-1;\gamma=+1$
$\leqslant l \leqslant n-1;$ $G_{1\beta}$	4	4	−4	4	−4	−4	4	−4	$-4\cos\{2lp\pi/(2n-1)\}$	$4\cos\{2lp\pi/(2n-1)\}$	$-4\cos\{2lp\pi/(2n-1)\}$	0	$4\cos\{2lp\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	0	0	0	
$E_{1\gamma}$	2	2	2	−2	−2	2	−2	−2	2	−2	−2	2	2	0	0	−2	0	0	0	0	0	0	0	0	
$E_{2\gamma}$	2	2	2	−2	−2	2	−2	−2	2	−2	−2	−2	2	0	0	2	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 2n-2;$ $G_{1\gamma}$ $\{G^+_{1\gamma}$	2	2	2	−2	−2	2	−2	−2	$2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$2i\sin\{l(2p-1)\pi/(2n-1)\}$	$2i\sin\{2lp\pi/(2n-1)\}$	$-2\cos\{l(2p-1)\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	$\alpha=+1;\beta=+1;\gamma=-1$
$G^-_{1\gamma}$	2	2	2	−2	−2	2	−2	−2	$2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$-2\cos\{2lp\pi/(2n-1)\}$	$2\cos\{l(2p-1)\pi/(2n-1)\}$	$2\cos\{2lp\pi/(2n-1)\}$	$-2i\sin\{l(2p-1)\pi/(2n-1)\}$	$-2i\sin\{2lp\pi/(2n-1)\}$	$-2\cos\{l(2p-1)\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 2n-1;$ $G_{1\alpha\beta}$ $\{G^+_{1\alpha\beta}$	2	−2	−2	2	−2	2	−2	2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)p\pi/(2n-1)\}$	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	$2\cos\{(2l-1)p\pi/(2n-1)\}$	$2i\sin\{(2l-1)(2p-1)\pi/(2n-1)\}$	$2i\sin\{(2l-1)(2p+1)\pi/(4n-2)\}$	$2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	0	0	0	0	0	0	0	0	$\alpha=-1;\beta=-1;\gamma=+1$
$G^-_{1\alpha\beta}$	2	−2	−2	2	−2	2	−2	2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)p\pi/(2n-1)\}$	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	$2\cos\{(2l-1)p\pi/(2n-1)\}$	$-2i\sin\{(2l-1)(2p-1)\pi/(2n-1)\}$	$-2i\sin\{(2l-1)(2p+1)\pi/(4n-2)\}$	$2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	0	0	0	0	0	0	0	0	
$G_{\alpha\gamma}$ $\{G^+_{\alpha\gamma}$	2	−2	2	−2	−2	−2	2	2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	2i	−2i	
$G^-_{\alpha\gamma}$	2	−2	2	−2	−2	−2	2	2	$2(-1)^p$	$2(-1)^{p+1}$	$2(-1)^{p+1}$	0	$2(-1)^p$	0	0	0	0	0	0	0	0	0	−2i	2i	$\alpha=-1;\ \beta=+1;\gamma=-1$
$\leqslant l \leqslant n-1;$ $G_{1\alpha\gamma}$	4	−4	4	−4	−4	−4	4	4	$4\cos\{(2l-1)p\pi/(2n-1)\}$	$-4\cos\{(2l-1)p\pi/(2n-1)\}$	$-4\cos\{(2l-1)p\pi/(2n-1)\}$	0	$4\cos\{(2l-1)p\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	0	0	0	
$G_{\beta\gamma}$ $\{G^+_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	−2	−2	2	0	2	0	0	0	0	0	2i	−2i	0	0	0	0	
$G^-_{\beta\gamma}$	2	2	−2	−2	2	−2	−2	2	−2	−2	2	0	2	0	0	0	0	0	−2i	2i	0	0	0	0	$\alpha=+1;\beta=-1;\gamma=-1$
$\leqslant l \leqslant n-1;$ $G_{1\beta\gamma}$	4	4	−4	−4	4	−4	−4	4	$-4\cos\{2lp\pi/(2n-1)\}$	$-4\cos\{2lp\pi/(2n-1)\}$	$4\cos\{2lp\pi/(2n-1)\}$	0	$4\cos\{2lp\pi/(2n-1)\}$	0	0	0	0	0	0	0	0	0	0	0	
$\leqslant l \leqslant 4n-2;$ $E_{l\alpha\beta\gamma}$	2	−2	−2	−2	2	2	2	−2	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$-2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)p\pi/(2n-1)\}$	$2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	$2\cos\{(2l-1)p\pi(2n-1)\}$	$2\sin\{(2l-1)(2p-1)\pi/(4n-2)\}$	$2\sin\{(2l-1)p\pi/(2n-1)\}$	$-2\cos\{(2l-1)(2p-1)\pi/(4n-2)\}$	0	0	0	0	0	0	0	0	$\alpha=-1;\beta=-1;\gamma=-1$
$D_{(4n-2)h}$	E											A^{2p-1}	A^{2p}	$A^{2p-1}BC$ $1 \leqslant p \leqslant 2n-1$	$A^{2p}BC$ $0 \leqslant p \leqslant 2n-2$		$A^{2q}C$ $0 \leqslant q \leqslant n$		$A^{4n-2q-1}C$ $1 \leqslant q \leqslant 2n-1$		$A^{4n-4-4q}B$ $0 \leqslant q \leqslant n-1$	$A^{4n-4q-2}B$ $1 \leqslant q \leqslant n-1$	$A^{4n-4q+1}B$ $1 \leqslant q \leqslant n$	$A^{4n-4q-1}B$ $1 \leqslant q \leqslant n-1$	$A^{4n-2}=B^2=C^2=E$ $BA=A^{4n-3}B$ $CA=AC$ $CB=BC$

TABLE 3 (*cont.*)

$\mathscr{G}_1(O_h)$	$1c_1$	$1c_2$	$1c_2$	$1c_2$	$6c_4$	$8c_6$	$8c_3$	$24c_4$	$12c_8$	$12c_8$	$6c_4$
	E	P^2	S^2	P^2S^2	P, P^3 Q, P^2Q PQ, P^3Q	PR, P^3QR^2 QR, P^3R^2 PQR, P^3QR^2 P^2R, P^2R^2	R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^2R	S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^3RS, P^3RS^3 P^2S, P^2S^3 P^2QS, P^2QS^3 $P^3R^2S, P^3R^2S^3$ $P^2R^2S, P^2R^2S^3$ P^3QRS, P^3QRS^3 RS, RS^3	PS, QS^3 P^3QS, P^3S^3 $PQRS, PRS^3$ P^3RS, P^3QRS^3 QR^2S, PQR^2S^3 $P^3QR^2S, P^3QR^2S^3$	QS, PS^3 P^3S, P^2QS^3 $PRS, PQRS^3$ P^3QRS, P^3RS^3 PQR^2S, QR^2S^3 $P^3QR^2S, P^3QR^2S^3$	PS^2, P^3S^2 QS^2, P^3QS^2 PQS^2, P^3QS^2
A_{1g}	1	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	1	1	1	−1	−1	−1	1
E_g	2	2	2	2	2	−1	−1	0	0	0	2
T_{1g}	3	3	3	3	−1	0	0	−1	1	1	−1
T_{2g}	3	3	3	3	−1	0	0	1	−1	−1	−1
A_{1u}	1	1	1	1	1	1	1	1	1	1	1
A_{2u}	1	1	1	1	1	1	1	−1	−1	−1	1
E_u	2	2	2	2	2	−1	−1	0	0	0	2
T_{1u}	3	3	3	3	−1	0	0	−1	1	1	−1
T_{2u}	3	3	3	3	−1	0	0	1	−1	−1	−1
G_α	4	−4	4	−4	0	2	−2	0	0	0	0
K^+_α	4	−4	4	−4	0	−1	1	0	0	0	0
K^-_α	4	−4	4	−4	0	−1	1	0	0	0	0
E_β	2	2	−2	−2	2	2	2	0	0	0	−2
G^+_β	2	2	−2	−2	2	−1	−1	0	0	0	−2
G^-_β	2	2	−2	−2	2	−1	−1	0	0	0	−2
I_β	6	6	−6	−6	−2	0	0	0	0	0	2
$E'_{\alpha\beta g}$	2	−2	−2	2	0	1	−1	0	$\sqrt{2}$	$-\sqrt{2}$	0
$E'_{\alpha\beta u}$	2	−2	−2	2	0	1	−1	0	$-\sqrt{2}$	$\sqrt{2}$	0
$E''_{\alpha\beta g}$	2	−2	−2	2	0	1	−1	0	$-\sqrt{2}$	$\sqrt{2}$	0
$E''_{\alpha\beta u}$	2	−2	−2	2	0	1	−1	0	$\sqrt{2}$	$-\sqrt{2}$	0
$G_{\alpha\beta g}$	4	−4	−4	4	0	−1	1	0	0	0	0
$G_{\alpha\beta u}$	4	−4	−4	4	0	−1	1	0	0	0	0
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	BD ACD ABC^2D	

$\mathscr{G}_1(O_h)$	$8c_6$	$8c_6$	$2c_2$	$2c_2$	$12c_4$	$8c_6$	$8c_6$	$8c_6$	$8c_6$	$24c_4$	$12c_8$	$12c_8$	192 elements
	$PRS^2, P^3QR^2S^2$ $QRS^2, P^3R^2S^2$ $PQRS^2, P^3QR^2S^2$ $P^3RS^2, P^3R^2S^2$	R^2S^2, RS^2 PR^2S^2, P^3QRS^2 QR^2S^2, P^3QRS^2 PQR^2S^2, P^3RS^2	T P^2S^2T	P^2T S^2T	PT, P^3T QT, P^3QT PQT, P^3QT PS^2T, P^3S^2T QS^2T, P^3QS^2T PQS^2T, P^3QS^2T	PRT QRT $PQRT$ P^2RT R^2S^2T PR^2S^2T QR^2S^2T PQR^2S^2T	P^3QR^2T P^3R^2T P^3QR^2T P^2R^2T RS^2T P^3QRS^2T P^3QRS^2T P^3RS^2T	R^2T PR^2T QR^2T PQR^2T PRS^2T QRS^2T $PQRS^2T$ P^2RS^2T	RT P^3QRT P^3QRT P^2RT $P^3QR^2S^2T$ $P^3R^2S^2T$ $P^3QR^2S^2T$ $P^2R^2S^2T$	ST, P^2ST $PQST, P^3QST$ R^2ST, P^2R^2ST PR^2ST, P^3R^2ST $QRST, P^3QRST$ P^2RST, RST S^3T, P^2S^3T PQS^3T, P^3QS^3T $R^2S^3T, P^2R^2S^3T$ $PR^2S^3T, P^3R^2S^3T$ QRS^3T, P^3QRS^3T P^2RS^3T, RS^3T	PST, P^3QST $PQRST, P^3RST$ QR^2ST, P^3QR^2ST QS^3T, P^3S^3T PRS^3T, P^3QRS^3T PQR^2S^3T, P^3QRS^3T	QST, P^3ST $PRST, P^3QRST$ PQR^2ST, P^3QR^2ST PS^3T, P^3QS^3T $PQRS^3T, P^3RS^3T$ $QR^2S^3T, P^3QR^2S^3T$	$P^4 = Q^4 = R^3 = S^4 = T^2 = E$ $P^2 = Q^2$ $QP = P^3Q$; $RP = QR$; $RQ = PQR$; $SP = P^3QS$; $SQ = P^3S$; $SR = R^2S$; $TP = PT$; $TQ = QT$; $TR = RT$; $TS = P^2S^2T$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	$\alpha = +1; \beta = +1$
A_{2g}	1	1	1	1	1	1	1	1	1	−1	−1	−1	
E_g	−1	−1	2	2	2	−1	−1	−1	−1	0	0	0	
T_{1g}	0	0	3	3	−1	0	0	0	0	−1	1	1	
T_{2g}	0	0	3	3	−1	0	0	0	0	1	−1	−1	
A_{1u}	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	
A_{2u}	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	1	
E_u	−1	−1	−2	−2	−2	1	1	1	1	0	0	0	
T_{1u}	0	0	−3	−3	1	0	0	0	0	1	−1	−1	
T_{2u}	0	0	−3	−3	1	0	0	0	0	−1	1	1	
G_α	2	−2	0	0	0	0	0	0	0	0	0	0	$\alpha = -1; \beta = +1$
K^+_α	−1	1	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0	
K^-_α	−1	1	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0	
E_β	−2	−2	0	0	0	0	0	0	0	0	0	0	$\alpha = +1; \beta = -1$
G^+_β	1	1	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0	
G^-_β	1	1	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0	
I_β	0	0	0	0	0	0	0	0	0	0	0	0	
$E'_{\alpha\beta g}$	−1	1	2	−2	0	1	1	−1	−1	0	$\sqrt{2}$	$-\sqrt{2}$	$\alpha = -1; \beta = -1$
$E'_{\alpha\beta u}$	−1	1	−2	2	0	−1	−1	1	1	0	$\sqrt{2}$	$-\sqrt{2}$	
$E''_{\alpha\beta g}$	−1	1	2	−2	0	1	1	−1	−1	0	$-\sqrt{2}$	$\sqrt{2}$	
$E''_{\alpha\beta u}$	−1	1	−2	2	0	−1	−1	1	1	0	$-\sqrt{2}$	$\sqrt{2}$	
$G_{\alpha\beta g}$	1	−1	4	−4	0	−1	−1	1	1	0	0	0	
$G_{\alpha\beta u}$	1	−1	−4	4	0	1	1	−1	−1	0	0	0	
O_h			I		AI BI ABI	ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$ $BA = AB$; $CA = BC$; $CB = ABC$ $DA = BD$; $DB = AD$; $DC = C^2D$; $IA = AI$; $IB = BI$; $IC = CI$; $ID = DI$.

TABLE 3 (*cont.*)

$\mathscr{R}_2(O_h)$	$1c_1$ E	$1c_2$ P^2	$1c_2$ T^2	$1c_2$ P^2T^2	$6c_4$ P, P^3 Q, P^2Q PQ, P^3Q	$8c_6$ PR, P^3QR^2 QR, P^2R^2 PQR, P^2QR^2 P^3R, P^2R^2	$8c_3$ R^2, R PR^2, P^3QR QR^2, P^2QR PQR^2, P^2R	$24c_2$ $S; ST^2$ $PQS; PQST^2$ $R^2S; R^2ST^2$ $PR^2S; PR^2ST^2$ $QRS; QRST^2$ $P^2RS; P^2RST^2$ $P^2S; P^2ST^2$ $P^3QS; P^3QST^2$ $P^2R^2S; P^2R^2ST^2$ $P^3R^2S; P^3R^2ST^2$ $P^3QRS; P^3QRST^2$ $RS; RST^2$	$12c_8$ PS P^2QS $PQRS$ P^3RS QR^2S P^3QR^2S PST^2 P^3QST^2 $PQRST^2$ P^3RST^2 QR^2ST^2 $P^3QR^2ST^2$	$12c_8$ QS P^2S PRS P^3QRS PQR^2S P^2QRS QST^2 P^2ST^2 $PRST^2$ P^3QRST^2 PQR^2ST^2 P^2QRST^2	$6c_4$ PT^2, P^3T^2 QT^2, P^2QT^2 PQT^2, P^3QT^2	$8c_6$ $PRT^2, P^3QR^2T^2$ $QRT^2, P^2R^2T^2$ $PQRT^2, P^2QR^2T^2$ $P^3RT^2, P^2R^2T^2$	$8c_6$ R^2T^2, RT^2 PR^2T^2, P^3QRT^2 QR^2T^2, P^2QRT^2 PQR^2T^2, P^2RT^2	$2c_4$ T, T^3	$2c_4$ P^2T, P^2T^3	$12c_4$ PT, P^3T^3 QT, P^2QT^3 PQT, P^3QT^3 PT^3, P^3T QT^3, P^2QT PQT^3, P^3QT	$8c_{12}$ $PRT, P^3QR^2T^3$ $QRT, P^2R^2T^3$ $PQRT, P^2QR^2T^3$ $P^3RT, P^2R^2T^3$	$8c_{12}$ PRT^3, P^3QR^2T QRT^3, P^2R^2T $PQRT^3, P^2QR^2T$ P^3RT^3, P^2R^2T	$8c_{12}$ R^2T, RT^3 PR^2T, P^3QRT^3 QR^2T, P^2QRT^3 PQR^2T, P^2RT^3	$8c_{12}$ R^2T^3, RT PR^2T^3, P^3QRT QR^2T^3, P^2QRT PQR^2T^3, P^2RT	$24c_2$ $ST; ST^3$ $PQST; PQST^3$ $R^2ST; R^2ST^3$ $PR^2ST; PR^2ST^3$ $QRST; QRST^3$ $P^2RST; P^2RST^3$ $P^2ST; P^2ST^3$ $P^3QST; P^3QST^3$ $P^2R^2ST; P^2R^2ST^3$ $P^3R^2ST; P^3R^2ST^3$ $P^3QRST; P^3QRST^3$ $RST; RST^3$	$12c_8$ PST P^2QST $PQRST$ P^3RST QR^2ST P^3QR^2ST PST^3 P^3QST^3 $PQRST^3$ P^3RST^3 QR^2ST^3 $P^3QR^2ST^3$	$12c_8$ QST P^2ST $PRST$ P^3QRST PQR^2ST P^2QRST QST^3 P^2ST^3 $PRST^3$ P^3QRST^3 PQR^2ST^3 P^2QRST^3	192 elements $P^4 = Q^4 = R^3 = S^2 = T^4 = E$ $P^2 = Q^2$ $QP = P^3Q$; $RP = QR$; $RQ = PQR$; $SP = P^3QS$; $SQ = P^3S$; $SR = R^2S$; $TP = PT$; $TQ = QT$; $TR = RT$; $TS = ST^3$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\alpha = +1; \beta = +1$
A_{2g}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	
E_g	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	2	2	2	−1	−1	−1	−1	0	0	0	
T_{1g}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	3	3	−1	0	0	0	0	−1	1	1	
T_{2g}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	3	3	−1	0	0	0	0	1	−1	−1	
A_{1u}	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	
A_{2u}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	1	
E_u	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	−2	−2	−2	1	1	1	1	0	0	0	
T_{1u}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	−3	−3	1	0	0	0	0	1	−1	−1	
T_{2u}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	−3	−3	1	0	0	0	0	−1	1	1	
$G_{\alpha g}$	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	4	−4	0	−1	−1	1	1	0	0	0	$\alpha = -1; \beta = +1$
$G_{\alpha u}$	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	−4	4	0	1	1	−1	−1	0	0	0	
$\mathscr{G}_{\alpha g}$ $\{G^+_{\alpha g}$	2	−2	2	−2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	−1	2	−2	0	1	1	−1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	
$\mathscr{G}_{\alpha g}$ $\{G^-_{\alpha g}$	2	−2	2	−2	9	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	−1	2	−2	0	1	1	−1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	
$\mathscr{G}_{\alpha u}$ $\{G^+_{\alpha u}$	2	−2	2	−2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	−1	−2	2	0	−1	−1	1	1	0	$-i\sqrt{2}$	$i\sqrt{2}$	
$\mathscr{G}_{\alpha u}$ $\{G^-_{\alpha u}$	2	−2	2	−2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	−1	−2	2	0	−1	−1	1	1	0	$i\sqrt{2}$	$-i\sqrt{2}$	
$E_{1\beta}$	2	2	−2	−2	2	2	2	0	0	0	2	−2	−2	0	0	0	0	0	0	0	0	0	0	$\alpha = +1; \beta = -1$
$E_{2\beta}$	2	2	−2	−2	2	−1	−1	0	0	0	2	1	1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0	
$E_{3\beta}$	2	2	−2	−2	2	−1	−1	0	0	0	2	1	1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0	
I_β	6	6	−6	−6	−2	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	
$G_{1\alpha\beta}$	4	−4	−4	4	0	2	−2	0	0	0	0	−2	2	0	0	0	0	0	0	0	0	0	0	$\alpha = -1; \beta = -1$
$G_{2\alpha\beta}$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0	
$G_{3\alpha\beta}$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0	
O_h	E				A B AB	AC AC^2 ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	BD ACD ABC^2D				I		AI BI ABI	ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$ $BA = AB$; $CA = BC$; $CB = ABC$; $DA = BD$; $DB = AD$; $DC = C^2D$; $IA = AI$; $IB = BI$; $IC = CI$; $ID = DI$

TABLE 3 (*cont.*)

$_3(O_h)$	$1e_1$ E	$1e_2$ P^2	$1e_2$ S^2	$1e_2$ P^2S^2	$6e_4$ P, P^3 Q, P^2Q PQ, P^3Q	$8e_3$ PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 P^3R, P^2R^2	$8e_3$ R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	$24e_4$ S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^3RS, P^2RS^3 P^2S, P^2S^3 P^2QS, P^2QS^3 $P^3R^2S, P^3R^2S^3$ $P^3R^2S, P^3R^2S^3$ P^3QRS, P^3QRS^3 RS, RS^3	$12e_8$ PS P^2QS $PQRS$ P^3RS QR^2S P^3QR^2S PS^3 P^3QS^3 $PQRS^3$ P^3RS^3 QR^3S^3 $P^3QR^3S^3$	$12e_8$ QS^3 P^3S^3 PRS^3 P^3QRS^3 PQR^2S $P^2QR^2S^3$ QS P^3S PRS P^3QRS PQR^3S P^2QR^3S	$6e_4$ PS^2, P^3S^2 QS^2, P^2QS^2 PQS^2, P^3QS^2	$8e_6$ $PRS^2, P^3QR^2S^2$ $QRS^2, P^3R^2S^2$ $PQRS^2, P^2QR^2S^2$ $P^3RS^2, P^2R^2S^2$	$8e_6$ R^2S^2, RS^2 PR^2S^2, P^3QRS^2 QR^2S^2, P^3QRS^2 PQR^2S^2, P^3RS^2	$2e_4$ T, S^2T	$2e_4$ P^2T, P^2S^2T	$12e_4$ PT, P^3S^2T QT, P^2QS^2T PQT, P^3QS^2T PS^2T, P^3T QS^2T, P^3QT PQS^2T, P^3QT	$8e_{12}$ $PRT, P^3QR^2S^2T$ $QRT, P^3R^2S^2T$ $PQRT, P^2QR^2S^2T$ $P^3RT, P^3R^2S^2T$	$8e_{12}$ PRS^2T, P^3QR^2T QRS^2T, P^3R^2T $PQRS^2T, P^2QR^2T$ P^3RS^2T, P^2R^2T	$8e_{12}$ R^2T, RS^2T PR^2T, P^3QRS^2T QR^2T, P^3QRS^2T PQR^2T, P^3RS^2T	$8e_{12}$ R^2S^2T, RT PR^2S^2T, P^3QRT QR^2S^2T, P^3QRT PQR^2S^2T, P^3RT	$24e_4$ ST, S^3T $PQST, PQS^3T$ R^2ST, R^2S^3T PR^2ST, PR^2S^3T $QRST, QRS^3T$ P^3RST, P^3RS^3T P^2ST, P^2S^3T P^2QST, P^2QS^3T $P^3R^2ST, P^3R^2S^3T$ $P^3R^2ST, P^3R^2S^3T$ P^3QRST, P^3QRS^3T RST, RS^3T	$12e_8$ PST P^2QST $PQRST$ P^3RST QR^2ST P^3QR^2ST PS^3T P^3QS^3T $PQRS^3T$ P^3RS^3T QR^3S^3T P^3QR^3ST	$12e_8$ QS^3T P^3S^3T P^3RS^3T P^3QRS^3T PQR^2S^3T $P^2QR^2S^3T$ QST P^3ST P^3RST P^2QRST PQR^3ST $P^2QR^3S^3T$	192 elements $P^4 = Q^4 = R^3 = S^4 = T^4 = E$ $P^2 = Q^2$; $S^2 = T^2$ $QP = P^3Q$; $RP = QR$; $RQ = PQR$ $SP = P^3QS$; $SQ = P^3S$; $SR = R^2S$ $TP = PT$; $TQ = QT$; $TR = RT$ $TS = S^3T$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\alpha = +1; \beta = +1$
A_{2g}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	
E_g	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	2	2	2	−1	−1	−1	−1	0	0	0	
T_{1g}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	3	3	−1	0	0	0	0	−1	1	1	
T_{2g}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	3	3	−1	0	0	0	0	1	−1	−1	
A_{1u}	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	
A_{2u}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	1	
E_u	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	−2	−2	−2	1	1	1	1	0	0	0	
T_{1u}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	−3	−3	1	0	0	0	0	1	−1	−1	
T_{2u}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	−3	−3	1	0	0	0	0	−1	1	1	
G'_{2g}	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	4	−4	0	−1	−1	1	1	0	0	0	$\alpha = -1; \beta = +1$
G'_{2u}	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	−4	4	0	1	1	−1	−1	0	0	0	
G^{+}_{2g}	2	−2	2	−2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	−1	2	−2	0	1	1	−1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	
G^{-}_{2g}	2	−2	2	−2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	−1	2	−2	0	1	1	−1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	
G^{+}_{2u}	2	−2	2	−2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	−1	−2	2	0	−1	−1	1	1	0	$-i\sqrt{2}$	$i\sqrt{2}$	
G^{-}_{2u}	2	−2	2	−2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	−1	−2	2	0	−1	−1	1	1	0	$i\sqrt{2}$	$-i\sqrt{2}$	
$E_{1\beta}$	2	2	−2	−2	2	2	2	0	0	0	−2	−2	−2	0	0	0	0	0	0	0	0	0	0	$\alpha = +1; \beta = -1$
$E_{2\beta}$	2	2	−2	−2	2	−1	−1	0	0	0	−2	1	1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0	
$E_{3\beta}$	2	2	−2	−2	2	−1	−1	0	0	0	−2	1	1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0	
I_{β}	6	6	−6	−6	−2	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	
$G_{1x\beta}$	4	−4	−4	4	0	2	−2	0	0	0	0	−2	2	0	0	0	0	0	0	0	0	0	0	$\alpha = -1; \beta = -1$
$G_{2x\beta}$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	0	0	0	
$G_{3x\beta}$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	0	0	0	
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	BD ACD ABC^2D				I		AI BI ABI	ACI BCI $ABCI$		C^2I AC^2I BC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$ $BA = AB$; $CA = BC$; $CB = ABC$ $DA = BD$; $DB = AD$; $DC = C^2D$; $IA = AI$; $IB = BI$; $IC = CI$; $ID = DI$

TABLE 3 (*cont.*)

$\mathscr{R}_4(O_h)$	$1c_1$	$1c_2$	$1c_2$	$1c_2$	$6c_4$	$8c_4$	$8c_3$	$24c_4$	$12c_8$	$12c_8$	$6c_4$	$8c_4$	$8c_6$	$2c_4$	$2c_4$	$12c_2$	$8c_{12}$	$8c_{12}$	$8c_{12}$	$8c_{12}$	$24c_4$	$12c_8$	$12c_8$	192 elements
	E	P^2	S^2	P^2S^2	P, P^3 Q, P^2Q PQ, P^3Q	PR, P^3QR^2 QR, P^3R^2 PQR, P^3QR^2 P^3R, P^3R^2	R^2, R PR^2, P^2QR QR^2, P^3QR PQR^2, P^3R	S, S^3 PQS, PQS^3 R^2S, R^2S^3 PR^2S, PR^2S^3 QRS, QRS^3 P^2RS, P^2RS^3 P^3S, P^3S^3 P^3QS, P^3QS^3 $P^3R^2S, P^3R^2S^3$ $P^3R^2S, P^3R^2S^3$ P^3QRS, P^3QRS^3 RS, RS^3	PS^2 P^3QS^2 $PQRS^2$ P^3RS^2 QR^2S^2 $P^3QR^2S^2$ PS P^3QS $PQRS$ P^3RS QR^2S P^3QR^2S	QS P^3S PRS P^3QRS PQR^2S P^3QR^2S QS^3 P^3S^3 PRS^3 P^3QRS^3 PQR^2S^3 $P^3QR^2S^3$	PS^2, P^3S^2 QS^2, P^2QS^2 PQS^2, P^3QS^2	$PRS^2, P^3QR^2S^2$ $QRS^2, P^3R^2S^2$ $PQRS^2, P^3QR^2S^2$ $P^2RS^2, P^2R^2S^2$	R^2S^2, RS^2 PR^2S^2, P^3QRS^2 QR^2S^2, P^3QRS^2 PQR^2S^2, P^3RS^2	S^3T T	P^2S^3T P^2T	PT QT PQT PS^2T QS^2T PQS^2T P^3T P^3QT P^2QT P^3S^2T P^2QS^2T P^3QS^2T	PRT QRT $PQRT$ P^2RT $P^2R^3S^3T$ $P^3R^3S^3T$ $P^3QR^3S^3T$ $P^3QR^3S^3T$	PQR^3T PR^3T QR^3T R^3T RS^3T P^3QRS^3T P^3QRS^3T P^2RS^3T	P^2R^3T P^3R^3T P^2QR^3T P^3QR^3T PRS^3T QRS^3T $PQRS^3T$ P^3RS^3T	RT P^3QRT P^3QRT P^3RT PQR^3S^3T PR^3S^3T QR^3S^3T R^3S^3T	ST, P^3ST $PQST, P^3QST$ R^3ST, P^3R^3ST PR^3ST, P^3R^3ST $QRST, P^3QRST$ P^3RST, RST S^3T, P^3S^3T PQS^3T, P^3QS^3T $R^3S^3T, P^3R^3S^3T$ $PR^3S^3T, P^3R^3S^3T$ QRS^3T, P^3QRS^3T P^3RS^3T, RS^3T	PST, P^3QST $PQRST, P^3RST$ QR^3ST, P^3QR^3ST PS^3T, P^3QS^3T $PQRS^3T, P^3RS^3T$ $QR^3S^3T, P^3QR^3S^3T$	QST, P^3ST $PRST, P^3QRST$ PQR^3ST, P^3QR^3ST QS^3T, P^3S^3T PRS^3T, P^3QRS^3T $PQR^3S^3T, P^3QR^3S^3T$	$P^4 = Q^4 = R^3 = S^4 = T^4 = E$ $P^2 = Q^2 = T^2$ $QP = P^3Q$; $RP = QR$; $RQ = PQR$ $SP = P^3QS$; $SQ = P^3S$; $SR = R^3S$ $TP = PT$; $TQ = QT$; $TR = RT$ $TS = S^3T$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
A_{2g}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	
E_g	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	2	2	2	−1	−1	−1	−1	0	0	0	
T_{1g}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	3	3	−1	0	0	0	0	−1	1	1	
T_{2g}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	3	3	−1	0	0	0	0	1	−1	−1	$\alpha = +1; \beta = +1$
A_{1u}	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	
A_{2u}	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	1	
E_u	2	2	2	2	2	−1	−1	0	0	0	2	−1	−1	−2	−2	−2	1	1	1	1	0	0	0	
T_{1u}	3	3	3	3	−1	0	0	−1	1	1	−1	0	0	−3	−3	1	0	0	0	0	1	−1	−1	
T_{2u}	3	3	3	3	−1	0	0	1	−1	−1	−1	0	0	−3	−3	1	0	0	0	0	−1	1	1	
G_α	4	−4	4	−4	0	2	−2	0	0	0	0	2	−2	0	0	0	0	0	0	0	0	0	0	
K_α^+	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0	$\alpha = -1; \beta = +1$
K_α^-	4	−4	4	−4	0	−1	1	0	0	0	0	−1	1	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0	
E_β	2	2	−2	−2	2	2	2	0	0	0	−2	−2	−2	0	0	0	0	0	0	0	0	0	0	
G_β^+	2	2	−2	−2	2	−1	−1	0	0	0	−2	1	1	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0	0	$\alpha = +1; \beta = -1$
G_β^-	2	2	−2	−2	2	−1	−1	0	0	0	−2	1	1	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0	0	
I_β	6	6	−6	−6	−2	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	
$G_{1\alpha\beta}^+$	2	−2	−2	2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	−1	1	2i	−2i	0	i	−i	i	−i	0	$\sqrt{2}$	$-\sqrt{2}$	
$G_{1\alpha\beta}^-$	2	−2	−2	2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	−1	1	−2i	2i	0	−i	i	−i	i	0	$\sqrt{2}$	$-\sqrt{2}$	
$G_{2\alpha\beta}^+$	2	−2	−2	2	0	1	−1	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	−1	1	2i	−2i	0	i	−i	i	−i	0	$-\sqrt{2}$	$\sqrt{2}$	$\alpha = -1; \beta = -1$
$G_{2\alpha\beta}^-$	2	−2	−2	2	0	1	−1	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	−1	1	−2i	2i	0	−i	i	−i	i	0	$-\sqrt{2}$	$\sqrt{2}$	
$K_{\alpha\beta}^+$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	4i	−4i	0	−i	i	−i	i	0	0	0	
$K_{\alpha\beta}^-$	4	−4	−4	4	0	−1	1	0	0	0	0	1	−1	−4i	4i	0	i	−i	i	−i	0	0	0	
O_h	E				A B AB	AC BC ABC	C^2, C AC^2 BC^2 ABC^2	D ABD C^2D AC^2D BCD CD	AD $ABCD$ BC^2D	BD ACD ABC^2D				I		AI BI ABI	ACI BCI $ABCI$		C^2I AC^2I ABC^2I	CI	DI $ABDI$ C^2DI AC^2DI $BCDI$ CDI	ADI $ABCDI$ BC^2DI	BDI $ACDI$ ABC^2DI	$A^2 = B^2 = C^3 = D^2 = I^2 = E$ $BA = AB$; $CA = BC$; $CB = ABC$ $DA = BD$; $DB = AD$; $DC = C^2D$; $IA = AI$; $IB = BI$; $IC = CI$; $ID = DI$

TABLE 3 (*cont.*)

	$1c_1$	$1c_2$	$30c_4$	$20c_6$	$20c_3$	$12c_5$	$12c_5$	$12c_{10}$	$12c_{10}$	120 elements
$\mathscr{R}(I)$	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q PV, P^3V PV^2, P^3V^2 PV^3, P^3V^3 PV^4, P^3V^4 PRV^2, P^3RV^2 $PR^2V^3, P^3R^2V^3$ $PQRV^4, P^3QRV^4$ $PQR^2V^2, P^3QR^2V^2$ RV, P^2RV $R^2V^4, P^2R^2V^4$ QRV^3, P^2QRV^3 QR^2V, P^2QR^2V	PR, P^2QR^2 QR, P^3R^2 PQR, P^3QR^2 PQR^2V, PR^2V^4 QV, R^2V^3 PQV^3, P^3RV^4 QR^2V^2, P^2QV^4 RV^3, P^3QRV^3 P^2R, P^2R^2 P^3QRV, P^3QV^3	R, R^2 PR^2, P^2QR PQR^2, P^3R QR^2, P^3QR $QV^4, P^2QR^2V^2$ PQV^2, QRV $PQRV^3, P^2RV^2$ PRV^4, P^3QV^3 $P^2QV, P^2R^2V^3$ $P^3R^2V^4, P^3QR^2V$	V, V^4 QV^3, P^2RV^4 $RV^3, P^2QR^2V^4$ $PQRV, P^3QV^3$ R^2V^2, QR^2V^3 P^3R^2V, P^3QRV^2	V^2, V^3 $PQV, P^3QR^2V^2$ QRV^2, PQR^2V^4 PRV^3, PR^2V P^3RV, P^2QRV^4 $P^3R^2V^3, P^3QV^4$	QV^2, P^3QRV $R^2V, PQRV^2$ RV^4, P^2QV^2 QR^2V^4, P^3RV^3 P^2V, P^2V^4 $P^3R^2V^3, P^3QR^2V^3$	PRV, QRV^4 PR^2V^3, PQV^4 PQR^2V^2, P^3QV P^2V^2, P^2V^3 $P^2QRV^2, P^3QR^2V^4$ P^3R^2V, P^3RV^3	$P^4 = Q^4 = R^3 = V^5 = E$ $Q^2 = P^2$ $QP = P^3Q$; $RP = QR$ $RQ = PQR$; $VP = PV^4$ $VQ = QR^2V^2$; $VR = P^2R^2V^4$
A	1	1	1	1	1	1	1	1	1	
T_1	3	3	-1	0	0	Φ	Φ^{-1}	Φ	Φ^{-1}	
T_2	3	3	-1	0	0	Φ^{-1}	Φ	Φ^{-1}	Φ	$\alpha = +1$
G	4	4	0	1	1	-1	-1	-1	-1	
H	5	5	1	-1	-1	0	0	0	0	
$E_{\frac{1}{2}}$	2	-2	0	1	-1	$-\Phi$	$-\Phi^{-1}$	Φ	Φ^{-1}	
$E_{\frac{7}{2}}$	2	-2	0	1	-1	Φ^{-1}	Φ	$-\Phi^{-1}$	$-\Phi$	$\alpha = -1$
$G_{\frac{3}{2}}$	4	-4	0	-1	1	-1	1	1	-1	
$I_{\frac{5}{2}}$	6	-6	0	0	0	1	-1	-1	1	
I	E		A B AB AF AF^2 AF^3 AF^4 ACF^2 AC^2F^3 $ABCF^4$ ABC^2F^3 CF C^2F^4 BCF^3 BC^2F	AC BC ABC ABC^2F, AC^2F^4 BF, C^2F^3 ABF^2 BC^2F^2 CF^2	C, C^2 AC^2 ABC^2 BC^2 BF^4 ABF^3, BCF $ABCF^3$ ACF^4	F, F^4 BF^2 CF^3 $ABCF$ C^2F^2, BC^2F^3	F^2, F^3 ABF BCF^2, ABC^2F^4 ACF^3, AC^2F	BF^3 $C^2F, ABCF^2$ CF^4 BC^2F^4	ACF, BCF^4 AC^2F^2, ABV^4 ABC^2F^2	$A^2 = B^2 = C^3 = F^5 = E$ $BA = AB$; $CA = BC$ $CB = ABC$; $FA = AF^4$ $FB = BC^2F^3$; $FC = C^2F^4$ $F^3C = BF$

($\Phi = \frac{1}{2}(1+\sqrt{5})$; $\Phi^{-1} = \frac{1}{2}(-1+\sqrt{5})$.)

TABLE 3 (*cont.*)

$\mathscr{R}_1(I_h) = \mathscr{R}(I) \times \{E, T\}$

	$1c_1$	$1c_2$	$30c_4$	$20c_6$	$20c_3$	$12c_5$	$12c_5$	$12c_{10}$	$12c_{10}$	$1c_4$	$1c_4$	$30c_2$	$20c_{12}$	$20c_{12}$	$12c_{20}$	$12c_{20}$	$12c_{20}$	$12c_{20}$	240 elements
$\mathscr{R}(I_h)$	E	P^2	P, P^3 Q, P^2Q PQ, P^3Q PV, P^3V PV^2, P^3V^2 PV^3, P^3V^3 PV^4, P^3V^4 PRV^2, P^3RV^2 $PR^2V^2, P^3R^2V^2$ $PQRV^4, P^3QRV^4$ $PQR^2V^3, P^3QR^2V^3$ RV, P^3RV $R^2V^4, P^2R^2V^4$ QRV^3, P^2QRV^3 QR^2V, P^2QR^2V	PR, P^3QR^2 QR, P^3R^2 PQR, P^2QR^2 PQR^2V, PR^2V^4 QV, R^2V^3 PQV^3, P^3RV^4 QR^2V^2, P^2QV^4 RV^2, P^3QRV^3 P^3R, P^3R^2 P^3QRV, P^3QV^2	R, R^2 PR^2, P^3QR PQR^2, P^3R QR^2, P^2QR $QV^4, P^2QR^2V^2$ PQV^3, QRV $PQRV^3, P^3RV^2$ PRV^4, P^3QV^3 $P^3QV, P^3R^2V^3$ $P^3R^2V^4, P^3QR^2V$	V, V^4 QV^2, P^3RV^4 $RV^3, P^3QR^2V^4$ $PQRV, P^3QV^3$ R^2V^3, QR^2V^3 P^3R^2V, P^3QRV^2	V^2, V^3 $PQV, P^3QR^2V^2$ QRV^3, PQR^2V^4 PRV^3, PR^2V P^3RV, P^2QRV^4 $P^3R^2V^3, P^3QV^4$	QV^3, P^3QRV $R^2V, PQRV^2$ RV^4, P^3QV^2 QR^2V^4, P^3RV^3 P^3V, P^3V^4 $P^3R^2V^2, P^2QR^2V^3$	PRV, QRV^4 PR^2V^3, PQV^4 PQR^2V^2, P^3QV P^3V^2, P^3V^3 $P^3QRV^2, P^3QR^2V^4$ P^3R^2V, P^3RV^3	T	P^2T	PT P^3T QT P^2QT PQT P^3QT PVT P^3VT PV^2T P^3V^2T PV^3T P^3V^3T PV^4T P^3V^4T PRV^2T P^3RV^2T PR^2V^2T $P^3R^2V^2T$ $PQRV^4T$ P^3QRV^4T PQR^2V^3T $P^3QR^2V^3T$ RVT P^3RVT R^2V^4T $P^3R^2V^4T$ QRV^3T P^3QRV^3T QR^2VT P^3QR^2VT	PRT QRT $PQRT$ PQR^2VT QVT PQV^3T QR^2V^2T RV^2T P^3RT P^3QRVT P^3QR^2T P^3R^2T P^2QR^2T PR^2V^4T R^2V^3T P^3RV^4T P^3QV^4T P^3QRV^3T P^2R^2T P^3QV^3T	PQR^2T PR^2T QR^2T $P^3R^2V^4T$ $P^3R^2V^3T$ PRV^4T QV^4T $PQRV^2T$ R^2T PQV^2T P^3RT P^3QRT P^2QRT P^3QR^2VT P^3QVT P^3QV^3T $P^3QR^2V^2T$ P^3RV^3T RT $QRVT$	VT V^4T QV^2T P^3RV^4T RV^3T $P^3QR^2V^4T$ $PQRVT$ P^3QV^3T R^2V^2T QR^2V^3T P^2R^2VT P^3QRV^2T	P^3V^4T P^3VT RV^4T P^3QV^2T QR^2V^4T P^3RV^3T QV^3T P^3QRVT $P^3QR^2V^3T$ $P^3R^2V^3T$ $PQRV^2T$ R^2VT	V^2T V^3T $PQVT$ $P^3QR^2V^2T$ QRV^2T PQR^2V^4T PRV^2T PR^2VT P^3RVT P^3QRV^4T $P^3R^2V^2T$ P^3QV^4T	P^3V^2T P^3V^3T PQR^2V^2T P^3QVT $P^3QR^2V^4T$ P^3QRV^2T P^3R^2VT P^3RV^2T QRV^4T $PRVT$ PQV^4T PR^2V^3T	$P^4 = Q^4 = R^3 = V^5 = T^4 = E$ $P^2 = Q^2 = T^2$ $QP = P^3Q$; $RP = QR$ $RQ = PQR$; $VP = PV^4$ $VQ = QR^2V^2$; $VR = P^3R^2V^4$ $TP = PT$; $TQ = QT$ $TR = RT$; $TV = VT$
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
T_{1g}	3	3	-1	0	0	Φ	Φ^{-1}	Φ	Φ^{-1}	3	3	-1	0	0	Φ	Φ^{-1}	Φ	Φ^{-1}	
T_{2g}	3	3	-1	0	0	Φ^{-1}	Φ	Φ^{-1}	Φ	3	3	-1	0	0	Φ^{-1}	Φ	Φ^{-1}	Φ	
G_g	4	4	0	1	1	-1	-1	-1	-1	4	4	0	1	1	-1	-1	-1	-1	
H_g	5	5	1	-1	-1	0	0	0	0	5	5	1	-1	-1	0	0	0	0	
A_u	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	$\alpha = +1$
T_{1u}	3	3	-1	0	0	Φ	Φ^{-1}	Φ	Φ^{-1}	-3	-3	1	0	0	$-\Phi$	$-\Phi^{-1}$	$-\Phi$	$-\Phi^{-1}$	
T_{2u}	3	3	-1	0	0	Φ^{-1}	Φ	Φ^{-1}	Φ	-3	-3	1	0	0	$-\Phi^{-1}$	$-\Phi$	$-\Phi^{-1}$	$-\Phi$	
G_u	4	4	0	1	1	-1	-1	-1	-1	-4	-4	0	-1	-1	1	1	1	1	
H_u	5	5	1	-1	-1	0	0	0	0	-5	-5	-1	1	1	0	0	0	0	
$G_{1\alpha}^{+}$	2	-2	0	1	-1	$-\Phi$	$-\Phi^{-1}$	Φ	Φ^{-1}	2i	-2i	0	i	$-$i	$-$iΦ	iΦ	$-$iΦ^{-1}	iΦ^{-1}	
$G_{1\alpha}^{-}$	2	-2	0	1	-1	$-\Phi$	$-\Phi^{-1}$	Φ	Φ^{-1}	-2i	2i	0	$-$i	i	iΦ	$-$iΦ	iΦ^{-1}	$-$iΦ^{-1}	
$G_{2\alpha}^{+}$	2	-2	0	1	-1	Φ^{-1}	Φ	$-\Phi^{-1}$	$-\Phi$	2i	-2i	0	i	$-$i	iΦ^{-1}	$-$iΦ^{-1}	iΦ	$-$iΦ	
$G_{2\alpha}^{-}$	2	-2	0	1	-1	Φ^{-1}	Φ	$-\Phi^{-1}$	$-\Phi$	-2i	2i	0	$-$i	i	$-$iΦ^{-1}	iΦ^{-1}	$-$iΦ	iΦ	
K_{α}^{+}	4	-4	0	-1	1	-1	-1	1	1	4i	-4i	0	$-$i	i	$-$i	$-$i	i	i	$\alpha = -1$
K_{α}^{-}	4	-4	0	-1	1	-1	-1	1	1	-4i	4i	0	i	$-$i	i	i	$-$i	$-$i	
O_{α}^{+}	6	-6	0	0	0	1	1	-1	-1	6i	-6i	0	0	0	i	i	$-$i	$-$i	
O_{α}^{-}	6	-6	0	0	0	1	1	-1	-1	-6i	6i	0	0	0	$-$i	$-$i	i	i	
I_h	E		A B AB AF AF^2 AF^3 AF^4 ACF^2 AC^2F^3 $ABCF^4$ ABC^2F^3 CF C^2F^4 BCF^3 BC^2F	AC BC ABC ABC^2F, AC^2F^4 BF, C^2F^3 ABF^2 BC^2F^2 CF^2	C, C^2 AC^2 ABC^2 BC^2 BF^4 ABF^3, BCF $ABCF^2$ ACF^4	F, F^4 BF^2 CF^2 $ABCF$ C^2F^2, BC^2F^3	F^2, F^3 ABF BCF^2, ABC^2F^4 ACF^3, AC^2F	BF^3 C^2F, $ABCF^2$ CF^4 BC^2F^4	ACF, BCF^4 AC^2F^2, ABF^4 ABC^2F^2	I		AI BI ABI AFI AF^2I AF^3I AF^4I ACF^2I AC^2F^2I $ABCF^4I$ ABC^2F^3I CFI C^2F^4I BCF^3I BC^2FI	ACI BCI $ABCI$ ABC^2FI BFI ABF^2I BC^2F^2I CF^2I AC^2F^4I C^2F^3I	ABC^2I AC^2I BC^2I ACF^4I BF^4I $ABCF^2$ C^2I ABF^3I CI $BCFI$	FI F^4I BF^2I CF^2I $ABCFI$ C^2F^2I BC^2F^3I	CF^4I BC^2F^4I BF^3I $ABCF^2I$ C^2FI	F^2I F^3I $ABFI$ BCF^2I ABC^2F^4I ACF^3I AC^2FI	ABC^2F^2I BCF^4I $ACFI$ ABF^4I AC^2F^2I	$A^2 = B^2 = C^3 = F^5 = I^2 = E$ $BA = AB$; $CA = BC$ $CB = ABC$; $FA = AF^4$ $FB = BC^2F^2$; $FC = C^2F^4$ $F^2C = BF$ $IA = AI$; $IB = B$. $IC = CI$; $IF = F_4$

($\Phi = \frac{1}{2}(1+\sqrt{5})$; $\Phi^{-1} = \frac{1}{2}(-1+\sqrt{5})$.)